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PLURISUBHARMONIC FUNCTIONS AND THE STRUCTURE OF COMPLETE KÄHLER MANIFOLDS WITH NONNEGATIVE CURVATURE

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Abstract

In this paper, we study global properties of continuous plurisubharmonic functions on complete noncompact Kähler manifolds with nonnegative bisectional curvature and their applications to the structure of such manifolds. We prove that continuous plurisubharmonic functions with reasonable growth rate on such manifolds can be approximated by smooth plurisubharmonic functions through the heat flow deformation. Optimal Liouville type theorem for the plurisubharmonic functions as well as a splitting theorem in terms of harmonic functions and holomorphic functions are established. The results are then applied to prove several structure theorems on complete noncompact Kähler manifolds with nonnegative bisectional or sectional curvature.

0. Introduction

In this paper, we are interested in the class of complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature. We shall first give a detailed study on the properties of heat flow with plurisubharmonic functions as initial data. Then we shall use the results to prove a Liouville theorem on plurisubharmonic functions and a splitting theorem related to harmonic and holomorphic functions. All these results will then be applied to obtain structure theorems on Kähler manifolds with nonnegative sectional or holomorphic bisectional curvature.

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One motivation of the present work is a program proposed by Yau [51, p. 622] on the study of parabolic manifolds: "The question is to demonstrate that every noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex euclidean space. If we only assume the nonnegativity of the bisectional curvature, the manifold should be biholomorphic to a complex vector bundle over a compact Hermitian symmetric space." As pointed out in [51], an important reason for this program comes from the celebrated results of Cheeger-Gromoll [5] and Gromoll-Meyer [16] on complete noncompact Riemannian manifolds with nonnegative or positive sectional curvature. It is also motivated by the work of Greene-Wu [13] on the Steinness of Kähler manifolds. In both cases, a key ingredient is to study Busemann functions.

In [5] it was proved that the Busemann function (with respect to all geodesic rays from a fixed point) on a complete noncompact Riemannian manifold with nonnegative sectional curvature is Lipschitz continuous, convex and is an exhaustion function. Then it was proved that a complete noncompact Riemannian manifold with nonnegative sectional curvature is diffeomorphic to the normal bundle over a compact totally geodesic submanifold without boundary, which is totally convex and is called the 'soul' of the manifold.

On a Kähler manifold with nonnegative holomorphic bisectional curvature, even though the Busemann function is no longer convex, it is still plurisubharmonic. This was proved by Wu [45]. Moreover, it is in fact strictly plurisubharmonic at the point where the holomorphic bisectional curvature is positive. Using this fact, it was proved by Greene-Wu that if the manifold has nonnegative sectional curvature and positive holomorphic bisectional curvature, then it is Stein because in this case the Busemann function is also an exhaustion function, see [12]-[15], [45]-[46] for more results. In the proof, it was first shown that a continuous *strictly* plurisubharmonic function can be approximated uniformly by a smooth one. The result of Grauert [10] can then be applied to conclude that the manifold is Stein.

If the manifold has nonnegative holomorphic bisectional curvature, then the Busemann function is only (continuous) plurisubharmonic instead of strictly plurisubharmonic. In order to use the Busemann function, it is desirable to approximate it by a smooth one. In general, it seems unlikely that a continuous plurisubharmonic function can be approximated by C^∞-plurisubharmonic functions. However, we shall prove the following rather general results on the solution of the heat equation with continuous plurisubharmonic function as initial data (see Theorem 3.1).

Theorem 0.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and let u be a continuous plurisubharmonic function on M satisfying*

$$|u|(x) \leq C \exp(ar^2(x)) \tag{0.1}$$

for some positive constants a, C, where $r(x)$ is the distance of x from a fixed point. Let v be the solution of the heat equation with initial data u. There exists $T_0 > 0$ depending only on a and there exists $T_0 > T_1 > 0$ such that the following are true:

(i) *For $0 < t \leq T_0$, $v(\cdot, t)$ is defined and is a smooth plurisubharmonic function.*

(ii) *Let*

$$\mathcal{K}(x,t) = \{w \in T_x^{1,0}(M) | \ v_{\alpha\bar{\beta}}(x,t)w^\alpha = 0, \text{ for all } \beta\}$$

be the null space of $v_{\alpha\bar{\beta}}(x,t)$. Then for any $0 < t < T_1$, $\mathcal{K}(x,t)$ is distribution on M. Moreover the distribution is invariant under parallel translations.

(iii) *If the holomorphic bisectional curvature is positive at some point, then $v(x,t)$ is strictly plurisubharmonic for all $0 < t < T_1$ for all $0 < t < T_1$ unless u is pluriharmonic.*

In particular, if u is a continuous plurisubharmonic function satisfying (0.1), then it can be approximated by smooth plurisubharmonic functions uniformly on compact subsets. In application, we shall make use of the properties of $v(x,t)$ in the above theorem rather than the result on approximation.

The proof of Theorem 0.1 relies on a general maximum principle for Hermitian symmetric (1,1) tensor η which satisfies a linear heat equation on a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. We obtain a maximum principle for η under some weak growth conditions on the rate of the average of $||\eta||$, the norm of η, over geodesic balls. Since there is no pointwise bound on $||\eta||$, we shall apply an indirect cutoff argument together with careful estimates on the solutions of the heat equation through extensive uses of the fundamental work of Li and Yau [27].

In [32], the first author raised the following question:

On a complete noncompact Kähler manifold with nonnegative Ricci curvature, is a plurisubharmonic function of sub-logarithmic growth a constant?

It is well-known that for the complex Euclidean space $\mathbb{C}^m$, the answer is positive. An affirmative answer to the above question is also a natural analogue, for plurisubharmonic functions, of Yau's Liouville theorem [49] for positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. An immediate application of the Theorem 0.1 is to give an affirmative answer to the above question on Kähler manifolds with nonnegative holomorphic bisectional curvature. Namely, we have the following (see Theorem 3.2):

Theorem 0.2. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let u be a continuous plurisubharmonic function on M. Suppose that*

$$\limsup_{x\to\infty}\frac{u(x)}{\log r(x)}=0.$$

Then u must be a constant.

Using Theorem 0.2 we obtain the following interesting results (see Theorem 4.1):

Theorem 0.3. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose f is a nonconstant harmonic function on M such that*

$$\limsup_{x\to\infty}\frac{|f(x)|}{r^{1+\epsilon}(x)}=0, \tag{0.2}$$

for any $\epsilon>0$, where $r(x)$ is the distance of x from a fixed point. Then f must be of linear growth and M splits isometrically as $\widetilde{M}\times\mathbb{R}$. Moreover the universal cover $\overline{M}$ of M splits isometrically and holomorphically as $\widetilde{M'}\times\mathbb{C}$, where $\widetilde{M'}$ is a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that there exists a nonconstant holomorphic function f on M satisfying (0.2). Then M itself splits as $\widetilde{M}\times\mathbb{C}$.

A well-known result in [49, 8] says that if the growth rate of a harmonic function on a complete noncompact Riemannian manifold with nonnegative Ricci curvature is 'close' to constant functions, namely if it is of sublinear growth, then it must be constant. Similar to this, the first result of Theorem 0.3 says that on a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature, if the growth rate of a harmonic function is 'close' to linear, then it must be of linear growth. On the other hand, for any $\delta > 0$, the 'round off' cones with metrics $dr^2 + r^2\, ds^2_{S^1(1/\sqrt{1+\delta})}$, where $S^1\left(\frac{1}{\sqrt{1+\delta}}\right)$ is the circle with radius $\frac{1}{\sqrt{1+\delta}}$, supports harmonic functions of growth $r^{1+\delta}(x)$.

One might also want to compare the splitting result in Theorem 0.3 with some previous related results in [4, 24, 3]. In [4], it was proved that if a complete noncompact Riemannian manifold with nonnegative Ricci curvature contains a line then a factor $\mathbb{R}$ can be splitted isometrically. In [24], it was proved that if a complete noncompact Kähler manifold with nonnegative Ricci curvature with complex dimension $m = n/2$ supports $n+1$ independent linear growth harmonic functions, then it is isometric and holomorphic to $\mathbb{C}^m$. In [3], Li's result was generalized to the Riemannian case, and the conclusion is that the manifold is isometric to Euclidean space. In [3], result on the splitting of the tangent cone in terms of linear growth harmonic functions was obtained.

The second part of the paper is to study the structure of complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature. The main tool is to use the heat flow with the Busemann functions as initial data. As mentioned above, the Busemann function is a continuous plurisubharmonic function on a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Hence Theorem 0.1 will be very useful. It turns out that Theorem 0.3 will be useful in the study too.

Before we state our next result, let us first introduce some conditions on a Kähler manifold. The first one is on the growth rate of volumes of geodesic balls. M is said to satisfy (VG$_k$) for $k > 0$, if there exists a constant $C > 0$ such that

$$V_o(r) \geq C r^k \tag{VG$_k$}$$

for all $r \geq 1$.

The other two conditions are on the decay of the curvature. Suppose M has nonnegative scalar curvature $\mathcal{R}$. M is said to satisfy the curvature decay condition (CD) if there exists a constant $C > 0$ (which might depend on o) such that

$$\fint_{B_o(r)} \mathcal{R} \leq \frac{C}{r} \tag{CD}$$

for all $r > 0$. M is said to satisfy the fast curvature decay condition (FCD) if there is a constant $C > 0$, so that

$$\text{(FCD)} \qquad \int_0^r s \left(\fint_{B_o(s)} \mathcal{R}(x)dx \right) ds \le C \log(r+2)$$

for all $r > 0$. (FCD) means that the average of the scalar curvature decays quadratically in the integral sense. Hence it is stronger than (CD). Our next result is the following splitting theorem (see Theorem 4.2):

Theorem 0.4. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature.*

(i) *Suppose M is simply connected, then $M = N \times M'$ holomorphically and isometrically, where N is a compact simply connected Kähler manifold, M' is a complete noncompact Kähler manifold and both N and M' have nonnegative holomorphic bisectional curvature. Moreover, M' supports a smooth strictly plurisubharmonic function with bounded gradient and satisfies* (VG_k) *and* (CD)*, where k is the complex dimension of M'. If, in addition, M has nonnegative sectional curvature outside a compact set, then M' is also Stein.*

(ii) *If the holomorphic bisectional curvature of M is positive at some point, then M itself supports a smooth strictly plurisubharmonic function with bounded gradient and satisfies* (VG_m) *and* (CD)*, where m is the complex dimension of M. If, in addition, M has nonnegative sectional curvature outside a compact set, then M is also Stein.*

The conclusion on the volume growth in the first statement in (ii) was first proved in [7] and the conclusion on curvature decay is a generalization of a result in [7]. The last statement in (ii) is a generalization of a result in [45]. Note that by [29, 19] (see also [2]), N in the theorem is a compact Hermitian symmetric manifold, but we shall not use this fact in the proof. Note also that N may not be present.

An immediate consequence of Theorem 0.4 is on the Steinness of complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature. Recall that a complete noncompact Riemannian manifold of dimension n with nonnegative Ricci curvature is said to have maximum volume growth if $V_x(r) \ge Cr^n$ for some positive constant C for all x and r. A result in [38] states that the Busemann function on a complete noncompact manifold with nonnegative Ricci curvature and with maximum volume growth is an exhaustion function. Using this, we prove as a corollary to Theorem 0.4 that a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and maximum volume growth is Stein. Here we assume neither that the holomorphic bisectional curvature is positive, which implies that the Busemann function is strictly plurisubharmonic, nor any curvature decay conditions as in [6]. We also prove the Steinness for the case that the manifold has a pole. This answers a question raised in [46, page 255] affirmatively. Recall that a Riemannian manifold is said to have a pole if there is a point p in the manifold such that the exponential map at p is a diffeomorphism.

To study M' (or M in Theorem 0.4(ii)) further, we obtain the following (see Theorem 4.3):

Theorem 0.5. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Assume that M supports a smooth strictly plurisubharmonic function u on M with bounded gradient.*

(i) *If M is simply connected, then $M = \mathbb{C}^\ell \times M_1 \times M_2$ isometrically and holomorphically for some $\ell \geq 0$, where M_1 and M_2 are complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that any polynomial growth holomorphic function on M is independent of the factor M_2, and any linear growth holomorphic function is independent of the factors M_1 and M_2. Moreover, M_1 supports a strictly plurisubharmonic function of logarithmic growth and satisfies* (FCD) *and* (VG_a)*, for any $a < k+1$, where $k = \dim_{\mathbb{C}} M_1$.*

(ii) *Suppose the holomorphic bisectional curvature of M is positive at some point, then either M has no nonconstant polynomial growth holomorphic function or M itself satisfies* (FCD) *and* (VG_a)*, for any $a < m+1$.*

There is an open question on whether the ring of polynomial growth holomorphic functions on a complete noncompact Kähler manifold with nonnegative curvature is finitely generated, see [52, p. 391]. By Theorems 0.4 and 0.5, in order to study polynomial growth holomorphic functions on a manifold with nonnegative holomorphic bisectional curvature which is either simply connected or has positive holomorphic bisectional curvature at some point, we may assume that M satisfies the fast curvature decay condition (FCD) and the volume growth condition (VG_a) for any $a < m+1$.

Together with the L^2 estimates [18] and the mean value inequality [25] Theorems 0.4 and 0.5 also imply that a simply connected complete noncompact Kähler manifold M with nonnegative holomorphic bisectional curvature supports many nontrivial holomorphic functions. Namely, M is a product of a compact Hermitian symmetric manifold, a complex Euclidean space, a complete manifold M_2 and a complete manifold M_1 such that each point of M_2 has local coordinate functions which are the restriction of global holomorphic functions with exponential growth of order ≤ 1 in the sense of Hadamard, and each point M_1 has local coordinate functions which are the restriction of global holomorphic functions with polynomial growth.

The results in the theorems on the decay rate of the average of the scalar curvature are related to the work of Shi [39] on the long time existence of the Kähler-Ricci flow, see also [51]. Theorems 0.4 and 0.5 also imply some uniformization type results when the volume growth of the manifold is small, see Corollary 4.3. Namely a simply-connected complete Kähler manifold with nonnegative bisectional curvature and slow volume growth is biholomorphic to the product of the complex line with a compact Hermitian symmetric manifold.

Next we shall study Kähler manifolds with nonnegative holomorphic bisectional curvature whose Busemann functions are exhaustion functions. We also assume that the universal cover of the manifold does not contain de Rham Euclidean factors. This class of manifolds contains manifolds which have nonnegative sectional curvature outside a compact set and positive Ricci curvature somewhere. Without assuming that the manifold is simply connected, one can describe the structure of M in a rather explicit way, see Theorem 5.1.

Theorem 0.6. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that the Busemann function is an exhaustion function. Suppose the universal cover $\widetilde{M}$ has no Euclidean factor. Then $\widetilde{M} = \widetilde{N} \times \widetilde{L}$ where $\widetilde{N}$ is a compact Hermitian symmetric manifold and $\widetilde{L}$ is Stein. Moreover, M is a holomorphic and Riemannian fibre bundle with fibre $\widetilde{N}$ over a Stein manifold $\widehat{M}$ with nonnegative holomorphic bisectional curvature such that $\widehat{M}$ is covered by $\widetilde{L}$.*

Using this structure result, Fangyang Zheng [54] proves that if in addition that M has nonnegative sectional curvature everywhere, M is in fact simply-connected and $M = N \times L$, where N is compact L is a Stein manifold and is diffeomorphic to R^{2l} where $l = \dim_{\mathbb{C}} L$. From this, one can prove that a complete noncompact Kähler manifold with nonnegative sectional curvature is a holomorphic and Riemannian fibre bundle over $\mathbb{C}^k/\Gamma$ for some discrete subgroup of the holomorphic isometry group of $\mathbb{C}^k$, with fibre $N \times L$ with the structures as above. The authors are grateful to Fangyang Zheng for allowing us to include his results and proofs in this work, see Theorem 5.2 and Corollary 5.1.

The results are motivated by the work of Takayama [43], where he proved that if M is a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and negative canonical line bundle and if M supports a continuous plurisubharmonic exhaustion function, then M has a structure of holomorphic fibre bundle over a Stein manifold whose fibre is biholomorphic to some compact Hermitian symmetric manifold. Obviously, our assumptions are stronger. However, in Theorem 0.6, the structure of the manifold is described more explicitly. Moreover, our proof is rather elementary and does not appeal to the result of [29] for example.

Finally, the methods of our study on the heat equation and the maximum principle can be applied to obtain the following result which is related Theorem 0.4 and in particular to the condition (FCD).

Theorem 0.7. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Then M is flat if*

$$\int_0^r s\left(⨍_{B_o(s)} \mathcal{R}(y)\,dy\right) ds = o(\log r)$$

provided that

$$\liminf_{r\to\infty}\left[\exp\left(-ar^2\right)\int_{B_o(r)} \mathcal{R}^2\right] < \infty$$

for some $a > 0$. where $\mathcal{R}$ is the scalar curvature of M.

For previous results in this direction, see [30, 32, 34, 6]. One of the main ideas is to solve the Poincaré-Lelong equation under rather weak conditions and then apply Theorem 0.2. The solution of the Poincaré-Lelong equation may have independent interest. See previous works [30, 34] on this problem.

Recently, Wu and Zheng [47]-[48] prove some interesting splitting results on Kähler manifolds with nonnegative or with nonpositive holomorphic bisectional curvature in terms of the rank of the Ricci tensor. In their works, the metric is assumed to be real analytic.

We organize the paper as follows: in §1 we study the solution of the heat equation; in §2 a maximum principle for Hermitian symmetric (1,1) tensor is given; we then apply the results to study the solution of the heat flow with continuous plurisubharmonic initial data in §3, a Liouville theorem for plurisubharmonic functions is also proved there; in §4–§6, we shall discuss the structure of Kähler manifolds with nonnegative holomorphic bisectional curvature; a solution to the Poincaré-Lelong equation will also be given in §6.

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1. Preliminary results

In this section, we shall derive some basic results on the solutions to the heat equation on a complete noncompact manifold with nonnegative Ricci curvature. These results will be used in later sections regularly. Specifically, we shall show that the Cauchy problem (1.6), which shall be defined in the following, can be solved, where only the average growth rate of the initial data over geodesic balls is assumed. This condition is useful in applications because in many cases a continuous plurisubharmonic function can only be approximated by a smooth function without point-wise estimations on the norms of the complex Hessians. However, Lemma 1.6 below shows that they can be estimated in the average sense. Corollary 1.1 and Lemma 1.4 will be used to keep track of the behaviors of the functions which approximate the Busemann function through the heat flow. These are important in the study of the structures of the manifolds.

We always assume that M^n is a complete noncompact Riemannian manifold with nonnegative Ricci curvature within this section. Let $H(x, y, t)$ be the heat kernel of M and let $o \in M$ be a fixed point. Denote the average of a function f over $B_x(r)$ by $\fint_{B_x(r)} f$. In this work, we shall make extensive uses of the fundamental work on the heat kernel estimates of Li and Yau [27]. We start with a L^p-estimate on the nonnegative solution to the heat equation.

Lemma 1.1. *Let $f \geq 0$ be a function on a complete noncompact Riemannian manifold M^n with nonnegative Ricci curvature and let*

$$u(x,t) = \int_M H(x,y,t) f(y) dy.$$

Assume that u is defined on $M \times [0,T]$ for some $T > 0$ and that for $0 < t \leq T$,

$$\lim_{r\to\infty} \exp\left(-\frac{r^2}{20t}\right) \int_{B_o(r)} f = 0. \tag{1.1}$$

Then for any $r^2 \geq t > 0$, and $p \geq 1$,

$$\begin{aligned} \fint_{B_o(r)} u^p(x,t) dx \leq C(n,p) \Bigg[& \fint_{B_o(4r)} f^p(x) dx \\ & + t^{-p} \left(\int_{4r}^{\infty} \exp\left(-\frac{s^2}{20t}\right) s \fint_{B_o(s)} f ds \right)^p \Bigg]. \end{aligned}$$

Proof. For any $p \geq 1$ and $r \geq \sqrt{t}$,

$$\begin{aligned} \int_{B_o(r)} u^p(x,t) dx &= \int_{B_o(r)} \left(\int_M H(x,y,t) f(y) dy \right)^p dx \\ &\leq C(p) \Bigg[\int_{B_o(r)} \left(\int_{B_o(4r)} H(x,y,t) f(y) dy \right)^p dx \\ &\quad + \int_{B_o(r)} \left(\int_{M \setminus B_o(4r)} H(x,y,t) f(y) dy \right)^p dx \Bigg]. \end{aligned} \tag{1.2}$$

Now for $x \in B_o(r)$ and $y \notin B_o(4r)$, we have $r(x,y) \geq 3/4 r(y)$. By the estimates of the heat kernel of Li and Yau [27, p. 176], we have:

(1.3)

$$
\begin{aligned}
&\int_{M\setminus B_o(4r)} H(x,y,t)f(y)dy\\
&\le C_1\int_{M\setminus B_o(4r)} \frac{1}{V_x(\sqrt{t})}\exp\left(-\frac{r^2(x,y)}{5t}\right)f(y)dy\\
&\le C_2\int_{M\setminus B_o(4r)} \frac{1}{V_x(r+\sqrt{t})}\cdot\left(\frac{r+\sqrt{t}}{\sqrt{t}}\right)^n\exp\left(-\frac{r^2(x,y)}{5t}\right)f(y)dy\\
&\le \frac{C_2}{V_o(\sqrt{t})}\cdot\left(\frac{r+\sqrt{t}}{\sqrt{t}}\right)^n\int_{M\setminus B_o(4r)}\exp\left(-\frac{r^2(x,y)}{5t}\right)f(y)dy\\
&\le \frac{C_2}{V_o(\sqrt{t})}\cdot\left(\frac{r+\sqrt{t}}{\sqrt{t}}\right)^n\int_{4r}^{\infty}\exp\left(-\frac{s^2}{10t}\right)\left(\int_{\partial B_o(s)} f\right)ds\\
&\le \frac{C_2}{10V_o(\sqrt{t})}\cdot\left(\frac{r+\sqrt{t}}{\sqrt{t}}\right)^n\int_{4r}^{\infty}\exp\left(-\frac{s^2}{10t}\right)\left(\int_{B_o(s)} f\right)d\left(\frac{s^2}{t}\right)\\
&\le C_3\left(\frac{r+\sqrt{t}}{\sqrt{t}}\right)^n\int_{4r}^{\infty}\frac{V_o(s)}{V_o(\sqrt{t})}\cdot\exp\left(-\frac{s^2}{10t}\right)\left(⨍_{B_o(s)} f\right)d\left(\frac{s^2}{t}\right)\\
&\le C_4t^{-1}\left[\int_{4r}^{\infty}\left(\frac{s}{\sqrt{t}}\right)^{2n}\exp\left(-\frac{s^2}{10t}\right)s⨍_{B_o(s)} f ds\right]\\
&\le C_5t^{-1}\left[\int_{4r}^{\infty}\exp\left(-\frac{s^2}{20t}\right)s⨍_{B_o(s)} f ds\right]
\end{aligned}
$$

for some constants C_1–C_5 depending only on n. Here we have used the volume comparison and the assumption (1.1) when we perform integration by parts in the fifth inequality.

On the other hand, by Hölder inequality and the fact that

$$\int_M H(x,y,t)dy = 1,$$

we have

$$\left(\int_{B_o(4r)} H(x,y,t)f(y)dy\right)^p \le \int_{B_o(4r)} H(x,y,t)f^p(y)dy.$$

Hence

$$
\begin{aligned}
(1.4) \qquad & \int_{B_o(r)} \left(\int_{B_o(4r)} H(x,y,t) f(y) dy \right)^p dx \\
&\le \int_{B_o(r)} \int_{B_o(4r)} H(x,y,t) f^p(y) dy\, dx \\
&\le \int_{B_o(4r)} f^p(y) \left(\int_{B_o(r)} H(x,y,t) dx \right) dy \\
&\le \int_{B_o(4r)} f^p(y) dy.
\end{aligned}
$$

The lemma follows from (1.2)–(1.4). q.e.d.

Let u be a continuous function on M such that

$$
(1.5) \qquad \fint_{B_o(r)} |u|(x)\, dx \le \exp(ar^2 + b)
$$

for some positive constant $a > 0$ and $b > 0$. Consider the following initial value problem

$$
(1.6) \qquad \begin{cases} \left(\Delta - \frac{\partial}{\partial t}\right) v(x,t) = 0 \\ \quad v(x,0) = u(x). \end{cases}
$$

Lemma 1.2. *The initial value problem (1.6) has a solution on* $M \times [0, \frac{1}{40a}]$. *Moreover, for* $(x,t) \in M \times (0, \frac{1}{40a}]$,

$$
v(x,t) = \int_M H(x,y,t) u(y) dy,
$$

where $H(x,y,t)$ *is the heat kernel of* M

Proof. For $j \ge 1$, let $0 \le \varphi_j \le 1$ be a smooth cutoff function such that $\varphi_j \equiv 1$ on $B_o(j)$ and $\varphi_j \equiv 0$ on $B_o(2j)$. Let $u_j = \varphi_j u$. Then u_j is continuous with compact support. Hence one can solve (1.6) with initial value u_j for all time. The solution v_j is given by

$$
(1.7) \qquad v_j(x,t) = \int_M H(x,y,t) u_j(y) dy
$$

for $(x,t)\in M\times(0,\infty)$. By Lemma 1.1, for any $0<t\le\frac{1}{40a}$ and for any $r\ge\sqrt{t}$,

(1.8)
$$
\begin{aligned}
\fint_{B_o(r)}|v_j|(x,t) &\le \fint_{x\in B_o(r)}\left(\int_M H(x,y,t)|u_j|(y)dy\right)dx\\
&\le C_1\left[\fint_{B_o(4r)}|u_j| + t^{-1}\int_{4r}^{\infty}\exp\left(-\frac{s^2}{20t}\right)s\fint_{B_o(s)}|u_j|ds\right]\\
&\le C_2\left[\fint_{B_o(4r)}|u_j| + e^b t^{-1}\int_{4r}^{\infty}\exp\left(-\frac{s^2}{20t}+as^2\right)sds\right]\\
&\le C_3e^b\left[\exp(16ar^2)+\int_{4r}^{\infty}\exp\left(-\frac{s^2}{40t}\right)d\left(\frac{s^2}{t}\right)\right]\\
&\le C_4e^b\left(\exp(16ar^2)+1\right)
\end{aligned}
$$

where C_1-C_4 are constants depending only on n. Since $|v_j|$ are sub-solutions of the heat equation and $|u_j|\le|u|$, by [26, Theorem 1.2] and (1.7) for $R^2\ge 1/(40a)$, we have that

$$
\sup_{B_o(\frac12 R)\times[0,\frac{1}{40a}]}|v_j|\le C_5\left[\exp(16aR^2+b)+\sup_{B_o(R)}|u|\right] \tag{1.9}
$$

for some constant C_5 depending only on n. From this, it is easy to see that after passing to a subsequence, v_j together their derivatives converge uniformly on compact sets on $M\times(0,\frac{1}{40a}]$ to a solution v of the heat equation. Moreover, for any $(x,t)\in M\times(0,\frac{1}{40a}]$, as in (1.3) we have

$$
\begin{aligned}
\left|\int_M H(x,y,t)u(y)dy - v_j(x,t)\right| &= \left|\int_M H(x,y,t)\left(u(y)-u_j(y)\right)dy\right|\\
&\le \int_{M\setminus B_o(j)}H(x,y,t)|u|(y)dy\\
&\le C_6\int_j^{\infty}\exp\left(-\frac{s^2}{20t}\right)s\fint_{\partial B_o(s)}|u|ds\\
&\le C_6\int_j^{\infty}\exp\left(-\frac{s^2}{40t}\right)d\left(\frac{s^2}{t}\right)\\
&\le C_6\int_{\frac{j^2}{t}}^{\infty}\exp\left(-\frac{1}{40}\tau\right)d\tau
\end{aligned}
$$

for some positive constant C_6. Here we have used the Harnack inequality [27, p. 168], the assumption (1.5) on u and the fact that $t \leq \frac{1}{40a}$. Hence it is easy to see that

$$v(x,t) = \int_M H(x,y,t)u(y)dy$$

and $v(x,0) = u(x)$. q.e.d.

In the next lemma, we shall obtain an estimate of the growth rate of $v(x,t)$ for fixed t in terms of the growth rate of u.

Lemma 1.3. *Let u and v be as in Lemma 1.2. Then for any $1 > \epsilon > 0$, there exists a constant $C = C(n,\epsilon,a,b)$ depending only on n, ϵ, a and b, and there exists $\frac{1}{40a} > T_0 > 0$ depending only on a and ϵ, such that for all $x \in M \times (0, T_0]$ with $r^2(x) \geq T_0$,*

$$\left| v(x,t) - \int_{B_x(\epsilon r)} H(x,y,t)u(y)dy \right| \leq C(n,\epsilon,a,b)$$

where $r = r(x)$.

Proof. Let $x \in M$ and let $r = r(x)$. It is easy to see that for $s \geq \epsilon r$

$$\fint_{B_x(s)} |u|(y)\,dy \leq C_1 \fint_{B_o((1+\epsilon^{-1})s)} |u|(y)\,dy$$

for some constant C_1 depending only on n and ϵ. Hence if $T_0 > 0$ is small enough, depending only on ϵ and a, then for $0 < t \leq T_0$, as before by [27, p. 176] we have

$$\begin{aligned}
&\int_{M\setminus B_x(\epsilon r)} H(x,y,t)|u|(y)dy \\
&\leq \frac{C_2}{V_x(\sqrt{t})} \int_{\epsilon r}^{\infty} \exp\left(-\frac{s^2}{5t}\right) \left(\int_{B_x(s)} |u|(y)\,dy \right) d\left(\frac{s^2}{t}\right) \\
&\leq C_3 \int_{\epsilon r}^{\infty} \left(\frac{s}{\sqrt{t}}\right)^n \exp\left(-\frac{s^2}{5t}\right) \left(\fint_{B_o((1+\epsilon^{-1})s)} |u|\,dy \right) d\left(\frac{s^2}{t}\right) \\
&\leq C_3 \int_{\epsilon r}^{\infty} \left(\frac{s}{\sqrt{t}}\right)^n \exp\left[-\frac{s^2}{5t} + a\left(1+\epsilon^{-1}\right)^2 s^2\right] d\left(\frac{s^2}{t}\right) \\
&\leq C_4
\end{aligned}$$

for some constants C_2, C_3 C_4 depending only on n, ϵ, a and b. From this the lemma follows. q.e.d.

Corollary 1.1. *With the same assumptions and notations as in Lemma 1.3, let $C(n,\epsilon,a,b)$ be the constant in the lemma. Then for $x \in M$ with $r = r(x) \geq \sqrt{T_0}$ such that $u \geq 0$ on $B_x(\epsilon r)$, then for any $0 \leq t < T_0$*

$$-C(n,\epsilon,a,b) + C_1 \inf_{B_x(\epsilon r)} u \leq v(x,t) \leq C(n,\epsilon,a,b) + \sup_{B_x(\epsilon r)} u$$

for some positive constant C_1 depending only on n and ϵ.

Proof. By Lemma 1.3, since $\int_M H(x,y,t)\,dy = 1$, we have

$$\begin{aligned} v(x,t) &\leq C(n,\epsilon,a,b) + \int_{B_x(\epsilon r)} H(x,y,t)u(y)\,dy \\ &\leq C(n,\epsilon,a,b) + \sup_{B_x(\epsilon r)} u. \end{aligned}$$

On the other hand, by the lower bound estimate of the heat kernel of Li-Yau [27, p. 182] and Lemma 1.3, we have that

$$\begin{aligned} v(x,t) &\geq -C(n,\epsilon,a,b) + \int_{B_x(\epsilon r)} H(x,y,t)u(y)\,dy \\ &\geq -C(n,\epsilon,a,b) + \frac{C_2}{V_x(\sqrt{t})} \int_{B_x(\epsilon\sqrt{t})} \exp\left(-\frac{r^2(x,y)}{5t}\right) u(y)\,dy \\ &\geq -C(n,\epsilon,a,b) + \frac{C_3 V_x(\epsilon\sqrt{t})}{V_x(\sqrt{t})} \inf_{B_x(\epsilon r)} u \\ &\geq -C(n,\epsilon,a,b) + C_4 \inf_{B_x(\epsilon r)} u \end{aligned}$$

for some positive constants C_2–C_4 depending only on n and ϵ. The proof of the corollary is completed. q.e.d.

Suppose u is Lipschitz, so that $|u(x) - u(y)| \leq \beta r(x,y)$, then v is defined for all t. We have the following.

Lemma 1.4. *Suppose u is Lipschitz so that $|u(x) - u(y)| \leq \beta r(x,y)$ for all $x,\ y \in M$ and let v be the solution of the heat equation with initial value u obtained in Lemma 1.2. Then for all $t > 0$,*

$$\sup_{x\in M} |\nabla v(x,t)| \leq \beta.$$

Proof. By [12, Proposition 2.1], for any $i > 0$, there is a smooth function u_i such that $\sup_M |\nabla u_i| \leq \beta + i^{-1}$ and $\sup_M |u_i - u| \leq i^{-1}$. By

Lemma 1.2, we can solve the initial value problem for the heat equation with initial value u_i. Denote the solution by v_i, which is defined for all t. Moreover,

$$|v - v_i|(x,t) \le \int_M H(x,y,t)|u(y) - u_i(y)|dy \le i^{-1}.$$

In particular, for $x \in M$ and $t > 0$, after passing to a subsequence if necessary,

$$\lim_{i\to\infty} |\nabla v_i|(x,t) = |\nabla v|(x,t). \tag{1.10}$$

However, using a more general version of [26, Proposition 2.4], see Lemma 1.5 below, we have

$$\sup_M |\nabla v_i|(x,t) \le \sup_M |\nabla u_i| \le \beta + i^{-1}. \tag{1.11}$$

The lemma follows from (1.10) and (1.11). q.e.d.

Lemma 1.5. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Let u be a smooth function on M with bounded gradient and let v be the solution of the heat equation initial value u. Then for any $t > 0$*

$$\sup_M |\nabla v(\cdot,t)| \le \sup_M |\nabla u|.$$

Proof. For any $T > 0$, by Lemma 1.3, since $|u|$ is of linear growth, we have

$$|v(x,t)| \le C_1\left(r(x) + 1\right)$$

for some C_1 for all $(x,t) \in M \times [0,T]$. On the other hand, using the fact that $\left(\Delta - \frac{\partial}{\partial t}\right) v^2 = 2|\nabla v|^2$, and using a suitable cut off function, one can obtain

$$\int_0^T \int_{B_o(r)} |\nabla v|^2 dxdt \le C_{18}\left[r^{-2}\int_0^T \int_{B_o(2r)} v^2 dxdt + \int_{B_o(2r)} u^2 dx\right]$$

and so

$$\int_0^T \int_M \exp\left(-r^2(x)\right)|\nabla v|^2 dxdt < \infty.$$

Combining with the fact that $|\nabla v|$ is a subsolution of the heat equation, the lemma follows from the maximum principle in [20] or [36, Theorem 1.2]. q.e.d.

Lemma 1.6. *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature. Assume that $g(x)$ is a smooth function satisfying*

$$\Delta g \geq f \tag{1.12}$$

for some continuous function $f(x)$. Assume that $f \geq -a$ for some $a \geq 0$ and there exists a monotone nondecreasing function $k(r)$ such that

$$g(x) \leq k(r(x)). \tag{1.13}$$

Then

$$\int_0^{\frac{1}{2}r} s \left(\fint_{B_o(s)} f_+(y)\, dy \right) ds \leq C(n) \left(k(5r) - g(o) + \frac{25}{2} a r^2 \right) \tag{1.14}$$

where $f_+ = \max\{f, 0\}$. In particular,

$$r^2 \fint_{B_o(r)} f_+ \leq C(n) \left(k(20r) - g(o) + 50ar^2 \right). \tag{1.15}$$

Proof. Let $M_1 = M \times \mathbb{R}$ and let $g_1(x,t) = g(x) + \frac{1}{2}at^2$ for $(x,t) \in M \times \mathbb{R}$. Then $\Delta_{M_1} g_1 \geq 0$. By Theorem 2.1 of [34], we have

$$\begin{aligned} C(n) \int_0^r s \left(\fint_{B_{o_1}(s)} \Delta_{M_1} g_1 \right) ds &\leq \sup_{B_{o_1}(5r)} g - g_1(o_1) \\ &\leq \sup_{B_o(5r)} g + \frac{25}{2} a r^2 - g(o) \end{aligned}$$

for some positive constant $C(n)$ depending only on n. Here $o_1 = (o, 0)$ and $B_{o_1}(s)$ is the geodesic ball in M_1 with center at o_1 and radius s. The lemma follows from the fact that $\Delta_{M_1} g_1 = \Delta g + a \geq f_+$ and Lemma 1.1 in [34]. q.e.d.

2. A maximum principle for tensors

In this section, we always assume that M^m is a complete noncompact Kähler manifold of complex dimension m (real dimension $n = 2m$). We denote the Kähler metric by $g_{\alpha\bar{\beta}}$. We want to establish a maximum principle for Hermitian symmetric $(1,1)$ tensor η satisfying the complex Lichnerowicz heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_{\gamma\bar{\delta}} = R_{\beta\bar{\alpha}\gamma\bar{\delta}}\eta_{\alpha\bar{\beta}} - \frac{1}{2}\left(R_{\gamma\bar{p}}\eta_{p\bar{\delta}} + R_{p\bar{\delta}}\eta_{\gamma\bar{p}}\right). \tag{2.1}$$

Assume $\eta(x,t)$ is defined on $M \times [0,T]$ for some $T > 0$. We also assume that there exists a constant $a > 0$ such that

$$\int_M \|\eta\|(x,0)\exp\left(-ar^2(x)\right)\,dx < \infty \tag{2.2}$$

and

$$\liminf_{r\to\infty}\int_0^T\int_{B_o(r)}\|\eta\|^2(x,t)\exp\left(-ar^2(x)\right)\,dx\,dt < \infty. \tag{2.3}$$

Here $\|\eta\|$ is the norm of $\eta_{\alpha\bar{\beta}}$ with respect to the Kähler metric. By (2.2), we have

$$\int_{B_o(r)} ||\eta||(x,0)\,dx \le \exp(ar^2)\cdot\mathcal{S} \tag{2.4}$$

where $\mathcal{S} = \int_M \|\eta\|(x,0)\exp\left(-ar^2(x)\right)\,dx$.

In the following, we always arrange the eigenvalues of η at a point in the ascending order.

Before we state our result, let us first fix some notations. Let $\varphi : [0,\infty) \to [0,1]$ be a smooth function so that $\varphi \equiv 1$ on $[0,1]$ and $\varphi \equiv 0$ on $[2,\infty)$. For any $x_0 \in M$ and $R > 0$, let $\varphi_{x_0,R}$ be the function on M defined by

$$\varphi_{x_0,R}(x) = \varphi\left(\frac{r(x,x_0)}{R}\right).$$

Let $f_{x_0,R}$ be the solution of

$$\left(\frac{\partial}{\partial t} - \Delta\right) f = -f$$

with initial value $\varphi_{x_0,R}$. Then $f_{x_0,R}$ is defined for all t and is positive and bounded for $t > 0$. In fact

$$f_{x_0,R}(x,t) = e^{-t}\cdot\int_M H(x,y,t)\varphi_{x_0,R}(y)dy.$$

We shall establish the following maximum principle.

Theorem 2.1. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let $\eta(x,t)$ be a Hermitian symmetric $(1,1)$ tensor satisfying (2.1) on $M \times [0,T]$ with $0 < T < \frac{1}{40a}$ such that $||\eta||$ satisfies (2.2) and (2.3). Suppose at $t = 0$, $\eta_{\alpha\bar{\beta}} \geq -bg_{\alpha\bar{\beta}}$ for some constant $b \geq 0$. Then there exists $0 < T_0 < T$ depending only on T and a so that the following are true:*

(i) *$\eta_{\alpha\bar{\beta}}(x,t) \geq -bg_{\alpha\bar{\beta}}(x)$ for all $(x,t) \in M \times [0,T_0]$.*

(ii) *For any $T_0 > t' \geq 0$, suppose there is a point x' in M^m and there exist constants $\nu > 0$ and $R > 0$ such that the sum of the first k eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\eta_{\alpha\bar{\beta}}$ satisfies*

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k \varphi_{x',R}$$

for all x at time t'. Then for all $t > t'$ and for all $x \in M$, the sum of the first k eigenvalues of $\eta_{\alpha\bar{\beta}}(x,t)$ satisfies

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k f_{x',R}(x, t - t').$$

Remark 2.1. It is well-known that the maximum principle for the heat equation is not true in general. The assumption of (2.3) type is the weakest and has been appeared for the scalar heat equation in [20], [36]. From this consideration, (2.3) is necessary. Also (2.2) in a sense ensures the solvability of the Cauchy problem of (2.1). Therefore, it is a reasonable assumption.

To prove the theorem, we begin with some lemmas. By Lemma 1.2 and (2.4), if we let

$$h(x,t) = \int_M H(x,y,t)\|\eta\|(y,0)\,dy,$$

then $h(x,t)$ is a solution of the heat equation defined on $M \times [0, \frac{1}{40a}]$ with initial value $||\eta||$. In the following T_0 always denotes a constant depending only on a and satisfying $T > T_0 > 0$. However, it may vary from line to line.

Lemma 2.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let η be a Hermitian symmetric $(1,1)$ tensor satisfying (2.1) on $M \times [0,T]$. Then $\|\eta\|(x,t)$ is a sub-solution of the heat equation. Moreover, if η also satisfies (2.2) and (2.3), then there exists $T > T_0 > 0$ depending only on a such that $\|\eta\|(x,t) \leq h(x,t)$ in $M \times [0,T_0]$.*

Proof. The first part is direct calculation. In fact using (2.1) one has

$$\left(\Delta - \frac{\partial}{\partial t}\right) \|\eta\|^2 = \|\eta_{\alpha\bar{\beta}s}\|^2 + \|\eta_{\alpha\bar{\beta}\bar{s}}\|^2 + 2R_{\alpha\bar{p}}\eta_{p\bar{\delta}}\eta_{\delta\bar{\alpha}} - 2R_{\alpha\bar{\beta}p\bar{q}}\eta_{\bar{p}q}\eta_{\bar{\alpha}\beta}$$
$$\geq \|\eta_{\alpha\bar{\beta}s}\|^2 + \|\eta_{\alpha\bar{\beta}\bar{s}}\|^2.$$

Combining with the observation

$$2|\nabla\|\eta\||^2 \leq \|\eta_{\alpha\bar{\beta}s}\|^2 + \|\eta_{\alpha\bar{\beta}\bar{s}}\|^2$$

we have that $\left(\Delta - \frac{\partial}{\partial t}\right)\|\eta\| \geq 0$.

Since $F = ||\eta|| - h$ is also a subsolution of the heat equation, the second conclusion follows from (2.3) and the proof of Theorem 1.2 of [36] because the positive part of F is less than or equal to $||\eta||$. q.e.d.

For any $r_2 > r_1$, let $A_o(r_1, r_2)$ denote the annulus $B_o(r_2) \setminus B_o(r_1)$. For any $R > 0$, let σ_R be a cut-off function which is 1 on $A_o(\frac{R}{4}, 4R)$ and 0 outside $A_o(\frac{R}{8}, 8R)$. We define

$$h_R(x,t) = \int_M H(x,y,t)\sigma_R(y)||\eta||(y,0)dy.$$

Then h_R satisfies the heat equation with initial data $\sigma_R||\eta||$.

Lemma 2.2. *Under the assumption (2.2) on η, there exists $T_0 > 0$ depending only on a such that the following are true:*

(i) *There exists a function $\tau = \tau(r) > 0$ with $\lim_{r\to\infty}\tau(r) = 0$ such that for all $R \geq \max\{\sqrt{T_0}, 1\}$ and for all $(x,t) \in A_o(\frac{R}{2}, 2R) \times [0, T_0]$,*

$$h(x,t) \leq h_R(x,t) + \tau(R).$$

(ii) *For any $r > 0$,*

$$\lim_{R\to\infty} \sup_{B_o(r)\times[0,T_0]} h_R = 0.$$

Proof. Note that h is defined on $M \times [0, \frac{1}{40a}]$, the first condition on T_0 is that $T_0 < \frac{1}{40a}$.

(i) Suppose $R^2 \geq \max\{T_0, 1\}$, where T_0 will be chosen later. For $0 < t < T_0$, by the definition of h and h_R, we have

$$h(x,t) \leq h_R(x,t) + \int_{M\setminus B_o(4R)} H(x,y,t)||\eta||(y,0)dy \tag{2.5}$$
$$+ \int_{B_o(\frac{R}{4})} H(x,y,t)||\eta||(y,0)dy.$$

For $x \in A_o(\frac{R}{2}, 2R)$ and $y \in B_o(\frac{R}{4})$, $r(x,y) \geq \frac{R}{4}$. Moreover, $V_x(R) \geq C(m)V_o(R)$ for some constant $C(m) > 0$ by the volume comparison. Hence if $x \in A_o(\frac{R}{2}, 2R)$, using (2.4), [27, p. 176] and volume comparison we have

$$
\begin{aligned}
(2.6) \qquad & \int_{B_o(\frac{R}{4})} H(x,y,t)||\eta||(y,0)dy \\
&\leq \sup_{y\in B_o(\frac{R}{4})} H(x,y,t) \int_{B_o(\frac{R}{4})} ||\eta||(y,0)\,dy \\
&\leq \frac{C_1}{V_x(\sqrt{t})} \sup_{y\in B_o(\frac{R}{4})} \exp\left(-\frac{r^2(x,y)}{5t}\right) \int_{B_o(\frac{R}{4})} ||\eta||(y,0)\,dy \\
&\leq \frac{C_2}{V_o(R)} \left(\frac{R}{\sqrt{t}}\right)^{2m} \exp\left(-\frac{R^2}{100t} + \frac{aR^2}{16}\right) \cdot \mathcal{S},
\end{aligned}
$$

for some constants C_1, C_2 depending only on m. On the other hand, since $x \in B_o(2R)$, then as in the proof of (1.3), if T_0 is small enough depending only on a, we have for $0 < t \leq T_0$

$$
\begin{aligned}
(2.7) \qquad & \int_{M\setminus B_o(4R)} H(x,y,t)||\eta||(y,0)\,dy \\
&\leq C_3 t^{-1} \left[\int_{4R}^{\infty} \exp\left(-\frac{s^2}{40t}\right) s \fint_{B_o(s)} ||\eta||(y,0)\,ds\right] \\
&\leq C_4 \mathcal{S} \int_{4R}^{\infty} \exp\left(-\frac{s^2}{40t} + as^2\right) d\left(\frac{s^2}{t}\right) \\
&\leq C_4 \mathcal{S} \int_{\frac{16R^2}{T_0}}^{\infty} \exp\left(-\frac{1}{80}\zeta\right) d\zeta
\end{aligned}
$$

where C_3–C_4 are constants depending only on m, provided T_0 is small enough depending only on a. Here we have used (2.4) and the fact $t \leq T_0$. From (2.5)–(2.7), (i) follows.

(ii) For $r > 0$ fixed, if T_0 is small enough depending only on a and

$(x,t) \in B_o(r) \times (0, T_0]$, for $R >> r$, as before we have

$$\begin{aligned} h_R(x,t) &\leq \int_{M \setminus B_o(\frac{R}{8})} H(x,y,t)||\eta||(y,0)dy \\ &\leq C_8 \int_{\frac{R}{8}}^{\infty} \exp\left(-\frac{s^2}{100t}\right) \fint_{B_o(s)} ||\eta||(y,0)ds \\ &\leq C_9 \mathcal{S} \int_{\frac{R}{8}}^{\infty} \exp\left(-\frac{s^2}{100t} + as^2\right) ds, \end{aligned}$$

for some constants C_8 and C_9 depending only on m. From this (ii) follows. q.e.d.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We only prove (ii) by assuming (i) is true. The proof of (i) is similar. Without loss of generality we assume that $t' = 0$. Let $T \geq T_0 > 0$ be small enough so that Lemmas 2.1, 2.2 are true. T_0 depends only on T and a. By Lemma 1.2 and Corollary 1.1, we can find a solution $\phi(x,t)$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi = \phi \tag{2.8}$$

such that $\phi(x,t) \geq \exp(c(r^2(x)+1))$ for some $c > 0$ for all $0 \leq t \leq T$. For example, let $\phi(x,t) = e^t \int_M H(x,y,t)h(y)dy$ with $h(y) \geq \exp(c'r^2)$ for some $c' > 0$.

Assume at $t = 0$, there exist $x_0 \in M$, $\nu > 0$ and $R > 0$ such that the first k eigenvalues $\lambda_1, \ldots, \lambda_k$ of $\eta_{\alpha\bar{\beta}}$ satisfy

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k \varphi_{x_0,R}$$

for all x at time $t = 0$. For simplicity, let us assume that $\nu = 1$.

By Lemma 2.1 and Lemma 2.2(i)

$$||\eta_{\alpha\bar{\beta}}||(x,t) \leq h(x,t) \leq h_R(x,t) + \tau(R) \tag{2.9}$$

for all $(x,t) \in \partial B_o(R) \times [0, T_0]$, where $\tau(R) > 0$ is a constant depending only on R and $\tau(R) \to 0$ as $R \to \infty$.

Let $\epsilon > 0$, for any $R > 0$, define $\psi = -f + \epsilon\phi + h_R + \tau(R) + b$, where h_R is the function defined above and $f(x,t) = f_{x_0,R}(x,t)$. Let $(\eta_R)_{\alpha\bar{\beta}} = \eta_{\alpha\bar{\beta}} + \psi g_{\alpha\bar{\beta}}$, where $g_{\alpha\bar{\beta}}$ is the metric tensor of M. Then at $t = 0$, at each point the sum of the first k eigenvalues of η_R is positive.

We want to prove that for any $T_0 \geq t > 0$ and $R > 0$, the sum of the first k eigenvalues of η_R in $B_o(R) \times [0, T_0]$ is positive, provided R is large enough.

Let R be large enough so that $\epsilon\phi - f > 0$ outside $B_o(R)$. Then by the definition of ψ and (2.9), $(\eta_R)_{\alpha\bar{\beta}}$ is positive definite on $\partial B_o(R) \times [0, T_0] \cup B_o(R) \times \{0\}$ and hence it is positive definite in a neighborhood of this set. Suppose there exists $(x, t) \in \overline{B}_o(R) \times [0, T_0]$ such that the sum of the first k eigenvalues of $(\eta_R)_{\alpha\bar{\beta}}$ is negative, then there exists $0 < t_1 \leq T_0$ and a point $x_1 \in \overline{B}_o(R)$ such that the sum of the first k eigenvalues of η_R at x_1 at time t_1 is zero but the sum of the first k eigenvalues of η_R at any point $(x, t) \in B_0(R) \times [0, t_1)$ is positive.

Let us fix the notations. Suppose $v_1, \ldots, v_m$ are unit eigenvectors of η_R at (x_1, t_1), with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. We may choose normal coordinates at x_1 such that $v_j = \frac{\partial}{\partial z^j}$ at x_1. In particular, if we write $v_j = v_j^\alpha \frac{\partial}{\partial z^\alpha}$, we have $v_j^\alpha = \delta_{\alpha j}$ at x_1. Note that the sum of the first k eigenvalues of a Hermitian form is the infimum of the traces of the form restricted to k-dimensional subspaces. Therefore $\sum_{\alpha,\beta=1}^k \left(g^{\alpha\bar{\beta}} (\eta_R)_{\alpha\bar{\beta}}\right) \geq 0$ for all (x, t) with $t \leq t_1$ and equals to zero at (x_1, t_1). Since η_R is positive definite in a neighborhood of $\partial B_o(R) \times [0, T_0]$, we conclude that x_1 is an interior point on $B_o(R)$.

Hence at (x_1, t_1), we have

$$0 \geq \left(\frac{\partial}{\partial t} - \Delta\right) \left(\sum_{\alpha,\beta=1}^k (\eta_R)_{\alpha\bar{\beta}} g^{\alpha\bar{\beta}}\right). \tag{2.10}$$

From now on repeated indices mean summation from 1 to m if there is no specification. Now

$$\frac{\partial}{\partial t} \left(\sum_{\alpha,\beta=1}^k (\eta_R)_{\alpha\bar{\beta}} g^{\alpha\bar{\beta}}\right) = \sum_{\alpha,\beta=1}^k \left(\frac{\partial}{\partial t} (\eta_R)_{\alpha\bar{\beta}}\right) g^{\alpha\bar{\beta}}. \tag{2.11}$$

Also at (x_1, t_1), we have

$$\Delta \left(\sum_{\alpha,\beta=1}^k (\eta_R)_{\alpha\bar{\beta}} g^{\alpha\bar{\beta}}\right) = \sum_{\alpha,\beta=1}^k \left(\Delta (\eta_R)_{\alpha\bar{\beta}}\right) g^{\alpha\bar{\beta}}. \tag{2.12}$$

By (2.10)–(2.12) and (2.1), at (x_1, t_1) we have,

$$0 \geq \sum_{\alpha,\beta=1}^{k} \Big[R_{\delta\bar{\gamma}\alpha\bar{\beta}} \left(\eta_{\gamma\bar{\delta}} + \psi g_{\gamma\bar{\delta}}\right) - \frac{1}{2} R_{\alpha\bar{p}} \left(\eta_{p\bar{\beta}} + \psi g_{p\bar{\beta}}\right) - \frac{1}{2} R_{p\bar{\beta}} \left(\eta_{\alpha\bar{p}} + \psi g_{\alpha\bar{p}}\right) \Big] g^{\alpha\bar{\beta}} + \sum_{\alpha,\beta=1}^{k} \bigg(\Big[\Big(\frac{\partial}{\partial t} - \Delta\Big) \psi \Big] g_{\alpha\bar{\beta}} - R_{\delta\bar{\gamma}\alpha\bar{\beta}} \psi g_{\gamma\bar{\delta}} + \frac{1}{2} \psi R_{\alpha\bar{p}} g_{p\bar{\beta}} + \frac{1}{2} \psi R_{\alpha\bar{p}} g_{p\bar{\beta}} \bigg) g^{\alpha\bar{\beta}}. \tag{2.13}$$

Since at (x_1, t_1), η has eigenvectors $v_p = \frac{\partial}{\partial z^p}$, for $1 \leq p \leq m$, with eigenvalue λ_p

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{k} \Big[R_{\delta\bar{\gamma}\alpha\bar{\beta}} \left(\eta_{\gamma\bar{\delta}} + \psi g_{\gamma\bar{\delta}}\right) - \frac{1}{2} R_{\alpha\bar{p}} \left(\eta_{p\bar{\beta}} + \psi g_{p\bar{\beta}}\right) - \frac{1}{2} R_{p\bar{\beta}} \left(\eta_{\alpha\bar{p}} + \psi g_{\alpha\bar{p}}\right) \Big] g^{\alpha\bar{\beta}} \\
&= \sum_{\alpha=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}} \lambda_\gamma - \sum_{\alpha=1}^{k} R_{\alpha\bar{\alpha}} \lambda_\alpha \\
&= \sum_{\alpha=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}} \lambda_\gamma - \sum_{\alpha=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}} \lambda_\alpha \\
&= \sum_{\alpha=1}^{k} \sum_{\gamma=k+1}^{m} \lambda_\gamma R_{\gamma\bar{\gamma}\alpha\bar{\alpha}} - \sum_{j=1}^{k} \sum_{\gamma=k+1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}} \lambda_\alpha \\
&= \sum_{\alpha=1}^{k} \sum_{\gamma=k+1}^{m} R_{\gamma\bar{\gamma}\alpha\bar{\alpha}} (\lambda_\gamma - \lambda_\alpha) \\
&\geq 0
\end{aligned} \tag{2.14}$$

where we have used that fact that M has nonnegative bisectional curvature, and $\lambda_\gamma \geq \lambda_\alpha$ for $\gamma \geq \alpha$. Also by (2.8), the definition of f and the fact that $\left(\frac{\partial}{\partial t} - \Delta\right) h_R = 0$, we have

$$\Big[\Big(\frac{\partial}{\partial t} - \Delta\Big) \psi \Big] = f + \epsilon\phi > 0. \tag{2.15}$$

Moreover

$$\left(-R_{\delta\bar{\gamma}\alpha\bar{\beta}}\psi g_{\gamma\bar{\delta}} + \frac{1}{2}\psi R_{\alpha\bar{p}} g_{p\bar{\beta}} + \frac{1}{2}\psi R_{\alpha\bar{p}} g_{p\bar{\beta}}\right) g^{\alpha\bar{\beta}} = 0. \tag{2.16}$$

From (2.13)–(2.16), we have a contradiction. Hence the sum of the first k eigenvalues of η_R is nonnegative for all $(x,t) \in B_o(R) \times (0, T_0]$. In particular, if $r > 0$ is fixed then the sum of the first k eigenvalues of η_R is nonnegative for all $(x,t) \in B_o(r) \times (0, T_0]$. Let $R \to \infty$, using Lemma 2.2, we conclude that the sum of the first k eigenvalues of

$$\eta_{\alpha\bar{\beta}}(x,t) + (-f(x,t) + \epsilon\phi(x,t) + b)\, g_{\alpha\bar{\beta}}(x,t)$$

is nonnegative on $B_o(r) \times [0, T_0]$ and hence on $M \times [0, T_0]$. Let $\epsilon \to 0$, we conclude that the sum of the first k eigenvalues of $\eta_{\alpha\bar{\beta}}(x,t)$ must be larger than or equal to $-kb + kf(x,t)$, for $0 < t \le T_0$ and for all $x \in M$. q.e.d.

From Theorem 2.1, applying the argument of [17] we have the following corollary.

Corollary 2.1. *Let M and η be as in Theorem 2.1 with $b = 0$. That is $\eta(x,0) \ge 0$ for all $x \in M$. Let $T_0 > 0$ be such that the conclusions of the theorem is true. For $0 < t < T_0$, let*

$$\mathcal{K}(x,t) = \{w \in T_x^{1,0}(M) |\ \eta_{\alpha\bar{\beta}}(x,t) w^\alpha = 0,\ \text{for all } \beta\}$$

be the null space of $\eta_{\alpha\bar{\beta}}(x,t)$. Then there exists $0 < T_1 < T_0$ such that for any $0 < t < T_1$, $\mathcal{K}(x,t)$ is a distribution on M. Moreover the distribution is invariant under parallel translations. In particular, if M is simply-connected, then $M = M_1 \times M_2$ isometrically and holomorphically, where $\mathcal{K}$ corresponds the tangent space of M_1, $(\eta_{\alpha\bar{\beta}}(x,t)) > 0$ on $M_2 \times (0, T_1)$. Both M_1 and M_2 are complete Kähler manifolds with nonnegative bisectional curvature.

Proof. By Theorem 2.1, $\eta(x,t) \ge 0$ on $M \times [0, T_0)$. By Theorem 2.1(ii), we conclude that if $\dim \mathcal{K}(x_0, t_0) \le k$ for some $x_0 \in M$ and $0 \le t_0 < T_0$ then $\dim \mathcal{K}(x,t) \le k$ for all $x \in M$ and $t > t_0$. It is easy to see that there exists $0 < T_1 < T$ such that $\dim \mathcal{K}(x,t)$ is constant on $M \times (0, T_1)$. Hence for each $0 < t < T_1$, $\mathcal{K}(x,t)$ is a smooth distribution on M. It remains to prove that the distribution is parallel for fixed t. We can proceed as in [17, Lemma 8.2].

Fix $0 < t_0 < T_1$, let $x_0 \in M$ and let $w_0 \in \mathcal{K}(x_0, t_0)$. Let $\gamma(\tau)$ be a smooth curve from x_0 and let $w(\tau)$ be the vector field obtained by parallel translation along γ. We want to prove that $w(\tau)$ is also in the null space $\mathcal{K}(\gamma(\tau), t_0)$ at $\gamma(\tau)$. Since the argument is local, we may assume that one can extend w to be a vector field in a neighborhood of $\gamma(\tau)$, and then extend w to be a vector field independent of time t. Now, projecting w onto $\mathcal{K}(x,t)$, we have a vector field v such that v is in $\mathcal{K}(x,t)$ for all x in a neighborhood of γ and for all t. The following computations are performed in a neighborhood of γ.

Since

$$\eta_{\alpha\bar{\beta}} v^{\alpha} = 0 \tag{2.17}$$

for all β, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left(\eta_{\alpha\bar{\beta}} v^{\alpha}\overline{v^{\beta}}\right) \\ &= \left(\frac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right) v^{\alpha}\overline{v^{\beta}} + \eta_{\alpha\bar{\beta}}\frac{\partial v^{\alpha}}{\partial t}\overline{v^{\beta}} + \eta_{\alpha\bar{\beta}} v^{\alpha}\frac{\partial \overline{v^{\beta}}}{\partial t} \\ &= \left(\frac{\partial}{\partial t}\eta_{\alpha\bar{\beta}}\right) v^{\alpha}\overline{v^{\beta}} \end{aligned} \tag{2.18}$$

where we have used (2.17). Choosing a unitary frame e_s at a point $\gamma(\tau)$, we have

$$\begin{aligned} 0 &= \Delta\left(\eta_{\alpha\bar{\beta}} v^{\alpha}\overline{v^{\beta}}\right) \\ &= \frac{1}{2}\left(\nabla_s\nabla_{\bar{s}} + \nabla_{\bar{s}}\nabla_s\right)\left(\eta_{\alpha\bar{\beta}} v^{\alpha}\overline{v^{\beta}}\right) \\ &= \left(\Delta\eta_{\alpha\bar{\beta}}\right) v^{\alpha}\overline{v^{\beta}} - \eta_{\alpha\bar{\beta}}\nabla_{\bar{s}} v^{\alpha}\nabla_s\overline{v^{\beta}} - \eta_{\alpha\bar{\beta}}\nabla_s v^{\alpha}\nabla_{\bar{s}}\overline{v^{\beta}} \end{aligned} \tag{2.19}$$

where we have used (2.17) so that

$$\left(\nabla_s\eta_{\alpha\bar{\beta}}\right) v^{\alpha} = -\eta_{\alpha\bar{\beta}}\nabla_s v^{\alpha},\ \left(\nabla_{\bar{s}}\eta_{\alpha\bar{\beta}}\right) v^{\alpha} = -\eta_{\alpha\bar{\beta}}\nabla_{\bar{s}} v^{\alpha}$$

and their complex conjugates.

Combining with (2.1), (2.18), (2.19), we have

$$0 = R_{t\bar{s}\alpha\bar{\beta}}\eta_{s\bar{t}} v^{\alpha}\overline{v^{\beta}} + 2\eta_{\alpha\bar{\beta}}\nabla_{\bar{s}} v^{\alpha}\nabla_s\overline{v^{\beta}} + 2\eta_{\alpha\bar{\beta}}\nabla_s v^{\alpha}\nabla_{\bar{s}}\overline{v^{\beta}}. \tag{2.20}$$

We may choose e_s so that at a point $\eta_{s\bar{t}} = a_s\delta_{st}$. Then

$$R_{t\bar{s}\alpha\bar{\beta}}\eta_{s\bar{t}} v^{\alpha}\overline{v^{\beta}} = R_{s\bar{s}\alpha\bar{\beta}} a_s v^{\alpha}\overline{v^{\beta}} = a_s R_{s\bar{s}v\bar{v}} \geq 0$$

because $a_s \geq 0$ and M has nonnegative bisectional curvature. Hence (2.20) and the fact that $\eta \geq 0$ imply that $\nabla_s v$ and $\nabla_{\bar{s}} v$ are in the null space $\mathcal{K}(\gamma(\tau), t_0)$.

Since $w(\tau)$ is parallel along $\gamma(\tau)$, and $w = v + w^{\perp}$, where $w^{\perp}$ is perpendicular to $\mathcal{K}(\gamma(\tau), t_0)$, we have

$$0 = \frac{D}{d\tau} w = \frac{D}{d\tau} v + \frac{D}{d\tau} w^{\perp}.$$

Hence

$$\frac{D}{d\tau} w^{\perp} = -\frac{D}{d\tau} v$$

which is in $\mathcal{K}$.

Now

$$\frac{d}{d\tau}\langle w^{\perp}, w^{\perp}\rangle = \left\langle \frac{D}{d\tau} w^{\perp}, w^{\perp}\right\rangle + \left\langle w^{\perp}, \frac{D}{d\tau} w^{\perp}\right\rangle = 0$$

because $\frac{D}{d\tau} w^{\perp}$ is in $\mathcal{K}$ and $w^{\perp}$ is perpendicular to $\mathcal{K}$. At $\gamma(0) = x_0$, $w = v_0$ and so $w^{\perp} = 0$ at $\gamma(0)$. Hence $w^{\perp} = 0$ for all τ and so w is in $\mathcal{K}$. The last statement follows from the De Rham decomposition. q.e.d.

Remark 2.2. Given the work [17] of Hamilton, the main difficulty for noncompact manifolds in the proof of Corollary 2.1 is to obtain the maximum principle Theorem 2.1. In particular, it can be proved more easily if we assume that η is bounded. In [1], results similar to the corollary are obtained independently for the case that $\eta = \mathrm{Ric}$ for the Kähler-Ricci flow in a complete noncompact Kähler manifolds with *bounded* nonnegative holomorphic bisectional curvature. However, it seems that a maximum principle is still needed in this case.

3. C^{∞}-approximation to continuous plurisubharmonic functions

In [12], it was proved that on a complete noncompact Kähler manifold a continuous *strictly* plurisubharmonic function can be approximated uniformly by C^{∞} strictly plurisubharmonic functions. If the function is only plurisubharmonic, then it can be approximated uniformly by C^{∞} functions whose complex Hessian are close to being nonnegative, see Lemma 3.2 below. In general, it seems unlikely that a continuous plurisubharmonic function can be approximated by C^{∞}-plurisubharmonic functions. However, in this section, we shall show that this can be done by the solution to the heat equation if the Kähler manifold has nonnegative holomorphic bisectional curvature, provided the continuous plurisubharmonic function satisfies a mild growth condition. Actually, we shall prove more in Theorem 3.1.

Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let u be a continuous plurisubharmonic function defined on M with growth rate satisfying

$$|u|(x) \le C \exp(ar^2(x)) \tag{3.1}$$

for some positive constants a and C. Let $v(x,t)$ be the solution to the heat equation on $M \times [0, \frac{1}{40a}]$ with initial value u, obtained by Lemma 1.2.

Theorem 3.1. *Let M^m, u and v be as above. There exists $T_0 > 0$ depending only on a and there exists $T_0 > T_1 > 0$ such that the following are true:*

(i) *For $0 < t \le T_0$, $v(\cdot, t)$ is a smooth plurisubharmonic function.*

(ii) *Let*

$$\mathcal{K}(x,t) = \{w \in T_x^{1,0}(M) |\ v_{\alpha\bar\beta}(x,t) w^\alpha = 0, \text{ for all } \beta\}$$

be the null space of $v_{\alpha\bar\beta}(x,t)$. Then for any $0 < t < T_1$, $\mathcal{K}(x,t)$ is a distribution on M. Moreover the distribution is invariant under parallel translations.

(iii) *If the holomorphic bisectional curvature is positive at some point, then $v(x,t)$ is strictly plurisubharmonic for all $0 < t < T_1$ unless u is pluriharmonic.*

As a corollary, we have:

Corollary 3.1. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and let u be a continuous plurisubharmonic function on M satisfying (3.1). Then there exist C^∞ plurisubharmonic functions u_i such that u_i converges to u uniformly on compact subsets of M. If in addition, the holomorphic bisectional curvature is positive at some point, then u_i can be chosen to be strictly plurisubharmonic unless u is pluriharmonic.*

We shall prove Theorem 3.1 by using the results in §2 together with a result in [12]. In order to use the results in §2, we need to the following estimates.

Lemma 3.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let u be a smooth function satisfying (3.1) and let v be the solution of the heat equation on $M \times [0, \frac{1}{40a}]$ with initial value u, obtained in Lemma 1.2. Moreover, assume that there exists $1 \geq b \geq 0$ such that*

$$u_{\alpha\bar{\beta}}(x) \geq -bg_{\alpha\bar{\beta}}(x) \tag{3.2}$$

for all $x \in M$. Let $||\rho||(x,t)$ be the norm of $v_{\alpha\bar{\beta}}(x,t)$. Then there exists $\frac{1}{40a} > T_0 > 0$ depending only on a with the following properties:

(i) *There exist constants C_1 and C_2, where C_2 depends only on a such that*
$$|v(x,t)| \leq C_1 \exp\left(C_2 r^2(x)\right).$$
for all $(x,t) \in M \times [0, T_0]$.

(ii) *There exist constants C_3 and C_4, where C_4 depends only on a such that*
$$⨍_{B_o(r)} ||\rho||(\cdot, 0) \leq C_3 \exp\left(C_4 r^2\right),$$
for all r.

(iii) *There exist constants C_5 and C_6, where C_6 depends only on a such that*
$$\int_0^{T_0} ⨍_{B_o(r)} ||\rho||^2(x,t)dxdt \leq C_5(1+T_0)\exp\left(C_6 r^2\right),$$
for all r.

Proof. In the following T_0 ($\frac{1}{40a} > T_0 > 0$) always denote a positive constant depending only on a, but its exact value may vary from line to line.

(i) By Lemma 1.3, we conclude that there exists $T_0 > 0$, such that if $r = r(x) \geq \sqrt{T_0}$ then

$$|v(x,t)| \leq \int_{B_x(\frac{r}{2})} H(x,y,t)|u|(y)dy + C_7$$

for all $(x,t) \in M \times (0, T_0]$ for some constant C_7 independent of x and t. Since u satisfies (3.1) and $\int_M H(x,y,t)dy = 1$, it is easy to see that (i) is true.

(ii) Let $f = \Delta u = g^{\alpha\bar{\beta}} u_{\alpha\bar{\beta}}$ and let $f = f_+ - f_-$, where f_+ (f_-) is the positive part (negative part) of f. Let $k^+(o,s) = \fint_{B_o(s)} f_+$. By the assumption on $u_{\alpha\bar{\beta}}$, $f^- \le mb \le m$. Applying (1.15) of Lemma 1.6 we have that

$$r^2 k^+(o,r) \le C(n)\left(\exp(100ar^2) - u(o) + 50mr^2\right). \tag{3.3}$$

Hence we have

$$r^2 k^+(o,r) \le C_{11}\exp(100ar^2) \tag{3.4}$$

for some constant C_{11} independent of r. On the other hand, at a point x, choose an normal coordinates such that $u_{\alpha\bar{\beta}} = \lambda_\alpha g_{\alpha\bar{\beta}}$. Since $u_{\alpha\bar{\beta}} \ge -bg_{\alpha\bar{\beta}}$ and $b \le 1$, for any α

$$\begin{aligned} -1 &\le -b \\ &\le \lambda_\alpha \\ &= \Delta u - \sum_{\beta\ne\alpha}\lambda_\beta \\ &\le f_+ + (m-1)b \\ &\le f_+ + (m-1). \end{aligned}$$

Therefore,

$$||\rho||(x) \le m\left(f_+(x) + (m-1)\right). \tag{3.5}$$

(ii) follows from (3.4) and (3.5).

(iii) By (i), there exists $\frac{1}{40a} > T_0 > 0$ such that for all $(x,t) \in M \times (0,T_0)$, we have

$$|v(x,t)| \le C_{12}\exp\left(C_{13}r^2(x)\right) \tag{3.6}$$

for some constants C_{12} independent of x and t, and C_{13} depending only on a. Using $\Delta u = f$, integrating by parts after multiplying a suitable cut-off function, one can prove that

$$\begin{aligned} \int_{B_o(r)}|\nabla u|^2 &\le C_{14}\left[r^{-2}\int_{B_o(2r)}u^2 + \int_{B_o(2r)}|u|\,|f|\right] \\ &\le C_{15}V_o(r)\left[\exp(8ar^2) + \exp(4ar^2)\fint_{B_o(2r)}|f|\right] \\ &\le C_{16}V_o(r)\exp(C_{17}ar^2) \end{aligned} \tag{3.7}$$

for some constants C_{14}–C_{16} independent of r, and C_{17} depending only on a. Here we have used (3.1), (ii) and the fact that $|f| \leq m||\rho||$. Using the fact that $\left(\Delta - \frac{\partial}{\partial t}\right) v^2 = 2|\nabla v|^2$, and multiplying a suitable cut off function, one can obtain

$$\begin{aligned} \int_0^{T_0} \fint_{B_o(r)} |\nabla v|^2 &\leq C_{18}\left[r^{-2} \int_0^{T} \fint_{B_o(2r)} v^2 + \fint_{B_o(2r)} u^2 \right] \\ &\leq C_{19}(T_0 + 1)\exp(C_{20} r^2) \end{aligned} \tag{3.8}$$

for some constants $C_{18} - C_{19}$ independent of r, and C_{20} depending only on a. Here we have used (3.1) and (3.6). By the Bochner formula,

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla v|^2 \geq 2|\nabla^2 v|^2.$$

Multiplying this inequality by a suitable cutoff function and integrating by parts, using (3.7) and (3.8) we have

$$\begin{aligned} \int_0^{T_0} \fint_{B_o(r)} |\nabla^2 v|^2 &\leq C_{21}\left[\frac{1}{r^2} \int_0^{T_0} \fint_{B_o(2r)} |\nabla v|^2 + \fint_{B_o(2r)} |\nabla u|^2 \right] \\ &\leq C_{22}(T_0 + 1)\exp(C_{23} r^2) \end{aligned}$$

for some constants $C_{21} - C_{22}$ independent of r, and constant C_{23} depending only on a. From this, (iii) follows. q.e.d.

We are ready to prove Theorem 3.1. We need the following approximation result of Greene-Wu [12, Corollary 2 to Theorem 4.1].

Lemma 3.2 (Greene-Wu)**.** *Let u be a continuous plurisubharmonic function on M. For any $b > 0$, there is a C^∞ function w such that*

(i) $|w - u| \leq b$ *on M; and*

(ii) $w_{\alpha\bar\beta} \geq -b g_{\alpha\bar\beta}$ *on M.*

Proof of Theorem 3.1. (i) Let u and v be as in the theorem. Choose $1 > \epsilon_i > 0$ such that $\epsilon_i \to 0$ as $i \to \infty$. By Lemma 3.2, we can find u_i such that

$$|u_i - u| \leq \epsilon_i \tag{3.9}$$

on M, and

$$(u_i)_{\alpha\bar\beta} \geq -\epsilon_i g_{\alpha\bar\beta} \tag{3.10}$$

on M. Since u satisfies (3.1), each u_i also satisfies (3.1). Namely,

$$|u_i|(x) \le c_i \exp(ar^2(x)) \tag{3.11}$$

for some constants c_i independent of x. By Lemma 1.2, we can solve the heat equation with initial data u_i on $M \times [0, \frac{1}{40a}]$. The solution is denoted by v_i. By Lemma 3.1, (3.10) and (3.11), there exist a constant $\frac{1}{40a} > T_0 > 0$ depending only on a such that

$$|v|(x,t) + |v_i|(x,t) \le d_i \exp(C_1 r^2(x)), \tag{3.12}$$

$$\fint_{B_o(r)} ||\rho_i||(\cdot, 0) \le d_i \exp\left(C_1 r^2\right), \tag{3.13}$$

and

$$\int_0^{T_0} \fint_{B_o(r)} ||\rho_i||^2(x,t)dxdt \le d_i(1+T_0)\exp\left(C_1 r^2\right) \tag{3.14}$$

for some constants d_i independent of r and for some constant C_1 depending only on a, where $||\rho_i||$ is the norm of $(v_i)_{\alpha\bar\beta}$. Here and below, $\frac{1}{40a} > T_0 > 0$ always denotes a constant depending only on a, but it may vary from place to place.

Since the complex Hessian $(v_i)_{\alpha\bar\beta}$ satisfies the Lichnerowicz heat equation (2.1) see [37, Lemma 2.1]. By (3.13), (3.14) and the maximum principle Theorem 2.1(i), there exists $\frac{1}{40a} > T_0 > 0$ such that

$$(v_i)_{\alpha\bar\beta}(x,t) \ge -\epsilon_i g_{\alpha\bar\beta}(x), \tag{3.15}$$

for all $(x,t) \in M \times [0, T_0]$.

By (3.12), we can apply the maximum principle of [20, 36], to conclude that

$$\sup_{M\times[0,T_0)} |v - v_i| \le \epsilon_i.$$

Hence passing to a subsequence if necessary v_i together with their derivatives subconverge to v uniformly on compact sets on $M \times (0, T_0)$. By (3.15), we conclude that $v_{\alpha\bar\beta}(x,t) \ge 0$ on $M \times (0, T_0)$.

(ii) Let T_0 be as in (3.15), which is obtained in Theorem 2.1. Let $T_0 > t_0 > 0$. Suppose there exists a point $x_0 \in M$ such that the sum of the first k eigenvalues of $v_{\alpha\bar\beta}(x_0, t_0)$ satisfies

$$\lambda_1 + \cdots + \lambda_k > 0,$$

then there exists $R > 0$ and $\nu > 0$ independent of i such that the sum of the first k eigenvalues of $(v_i)_{\alpha\bar{\beta}}(x, t_0)$ satisfies:

$$\lambda_{i,1} + \cdots + \lambda_{i,k} > k\nu,$$

on $B_{x_0}(2R)$. Since $(v_i)_{\alpha\bar{\beta}}$ satisfies (3.15), the sum of the first k eigenvalues of $(v_i)_{\alpha\bar{\beta}}$ satisfies:

$$\lambda_{i,1} + \cdots + \lambda_{i,k} > -k\epsilon_i + k\nu\varphi_{x_0,R}$$

at every point $x \in M$ at time t_0, where $\varphi_{x_0,R}$ is the nonnegative function as in Theorem 2.1. By Theorem 2.1(ii), for $T_0 > t > t_0$, the sum of the first k eigenvalues of $(v_i)_{\alpha\bar{\beta}}$ at (x, t) satisfies:

$$\lambda_{i,1} + \cdots + \lambda_{i,k} \geq -k\epsilon_i + k\nu f_{x_0,R}(x, t - t_0).$$

where $f_{x_0,R}$ is the function defined in Theorem 2.1. Note that $f_{x_0,R}(x, s) > 0$ if $s > 0$. Let $i \to \infty$, we conclude that the sum of the first k eigenvalues of $v_{\alpha\bar{\beta}}$ at (x, t) satisfies

$$\lambda_1 + \cdots + \lambda_k \geq k\nu f_{x_0,R}(x, t - t_0).$$

Hence we have proved that if there exists a point $x_0 \in M$ such that the sum of the first k eigenvalues of $v_{\alpha\bar{\beta}}(x_0, t_0)$ is positive, then for all $x \in M$ and $t > t_0$, the sum of the first k eigenvalues of $v_{\alpha\bar{\beta}}(x, t)$ is also positive. One can then proceed as in the proof of Corollary 2.1 to conclude that (ii) is true.

(iii) Suppose M has positive holomorphic bisectional curvature at x_0. By the proof of Corollary 2.1, there exists $0 < T_1 < T_0$ such that $\dim \mathcal{K}$ is constant on $M \times (0, T_1)$. For $0 < t < T_1$, suppose $\dim \mathcal{K}(x_0, t) > 0$. If $\dim \mathcal{K}(x_0, t) < m$, then by (ii) locally near x_0, M can be splitted isometrically as a nontrivial product of two Kähler manifold with nonnegative holomorphic bisectional curvature. This is impossible. If $\dim \mathcal{K}(x_0, t) = m$, then $\dim \mathcal{K} \equiv m$ on $M \times (0, T_1)$. This implies that $v(\cdot, t)$ is pluriharmonic for $0 < t < T_1$. Since $v(x, t) \to u(x)$ uniformly on compact sets as $t \to 0$, u is smooth and pluriharmonic. Hence if u is not pluriharmonic, then $\dim \mathcal{K}(x_0, t) = 0$ and so $\dim \mathcal{K}(x, t) = 0$ for all x. This implies that $v(\cdot, t)$ is strictly plurisubharmonic. The proof of the theorem is completed. q.e.d.

Using Theorem 3.1, we shall prove the following Liouville theorem which will be used to prove a splitting theorem as well as a gap theorem in Sections 4 and 6.

Theorem 3.2. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let u be a continuous plurisubharmonic function on M. Suppose that*

$$\limsup_{x\to\infty}\frac{u(x)}{\log r(x)}=0. \tag{3.16}$$

Then u must be a constant.

To prove the theorem we need the following lemma.

Lemma 3.3 ([32, Proposition 4.1])**.** *Let M^m be a complete noncompact Kähler manifold of complex dimension m, with nonnegative Ricci curvature. Let $u(x)$ be a plurisubharmonic function on M satisfying (3.16). Then $(\partial\bar{\partial}u)^m=0$.*

Proposition 4.1 stated in [32] is under the assumption that M is nonparabolic. However, the proof without any changes also works for general complete Kähler manifolds with nonnegative Ricci curvature.

Proof of Theorem 3.2. Let M and u satisfy the conditions in Theorem 3.2. Let $\widetilde{M}$ be the universal cover of M, then the distance function in $\widetilde{M}$ dominates the distance function in M. Hence $\widetilde{M}$ and the lift $\widetilde{u}$ of u also satisfy the conditions in the theorem. Therefore, we may assume that M is simply connected.

First we let $u_c=\max\{u,c\}$. By the assumption (3.16) it is easy to see that u_c satisfying (3.1) and u_c is plurisubharmonic. Therefore, we can solve the heat equation with $u_c(x)$ as the initial data. Denote the solution by v_c on $M\times[0,T_0]$. By adding a constant we can also assume that $u_c(x)\geq 0$. Applying Theorem 3.1(i) to $v_c(x,t)$ we conclude that $v_c(x,t)$ is plurisubharmonic. By Theorem 3.1(ii), for any $t_0>0$ small enough, $M=M_1\times M_2$ isometrically and holomorphically such that $(v_c)_{\alpha\bar{\beta}}$ is zero when restricted on M_1 and $(v_c)_{\alpha\bar{\beta}}$ is positive everywhere when restricted on M_2 by the De Rham decomposition (cf. Theorem 8.1, page 172 of [22]). By Corollary 1.1, we still have

$$\limsup_{x\to\infty}\frac{v_c(x,t_0)}{\log r(x)}=0. \tag{3.17}$$

Hence when restricted on M_2, (3.17) is still true. This contradicts Lemma 3.3 and the fact that $(v_c)_{\alpha\bar{\beta}}$ is positive when restricted on M_2, unless $M=M_1$. Hence $(v_c)_{\alpha\bar{\beta}}(x,t_0)\equiv 0$ on M for all $0<t_0$ small enough. By the gradient estimate of Cheng-Yau [8] and (3.17) we can conclude that that $v_c(x,t_0)$ is a constant, provided $t_0>0$ is small enough. Hence u_c is a constant. Since c is arbitrary, it shows that $u(x)$ is also a constant. q.e.d.

4. Structure of nonnegatively curved Kähler manifolds I

In this and the next section, we shall apply the results in the previous two sections to study the structure of complete noncompact Kähler manifolds with nonnegative sectional or holomorphic bisectional curvature. Let us begin with some lemmas.

Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Recall the definition of the Busemann function at a point $o \in M$, see [5]. Let γ be a ray from o parametrized by arc length. Then the Busemann $\mathcal{B}_\gamma(x)$ is defined as

$$\mathcal{B}_\gamma(x) = \lim_{s\to\infty} (s - d(x, \gamma(s)))$$

and the Busemann function $\mathcal{B}$ is defined as

$$\mathcal{B}(x) = \sup_\gamma \mathcal{B}_\gamma(x)$$

where the supremum is taken over all rays from o. It is well-known that $\mathcal{B}$ is Lipschitz with Lipschitz constant 1. Since M has nonnegative holomorphic bisectional curvature, by the result of Wu [45, p. 58], $\mathcal{B}$ is a continuous plurisubharmonic function on M. Let $v(x,t)$ be the solution of the heat equation with initial value $\mathcal{B}$. Then v is defined for all t. We collect some facts in the follow lemma for easy reference.

Lemma 4.1. *With the above assumptions and notations, the following are true:*

(i) *For any $t > 0$, $v(\cdot,t)$ is a smooth plurisubharmonic function. If the holomorphic bisectional curvature of M is positive at some point, then $v(\cdot,t)$ is strictly plurisubharmonic, unless $\mathcal{B}$ is pluriharmonic.*

(ii) *For any $t > 0$,*
$$\sup_M |\nabla v(\cdot,t)| \le 1.$$

(iii) *For any $t > 0$, $v(\cdot,t)$ grows linearly when restricted on a ray from o. If in addition, $\mathcal{B}$ is an exhaustion function of M, then $v(\cdot,t)$ is also an exhaustion function of M for all $t > 0$.*

(iv) *There exists* $T_0 > 0$*, such that for any* $0 < t < T_0$*, the null space* $\mathcal{K}(x,t) \subset T_x^{(1,0)}(M)$ *of* $v_{\alpha\bar{\beta}}(x,t)$ *is a parallel distribution on* M*.*

Proof. (i) and (iv) are just special cases of Theorem 3.1. Note that from the proof of Theorem 3.1, (i) is true for any $t > 0$. (ii) follows from Lemma 1.4. It remains to prove (iii). Let γ be a ray from o and let $x = \gamma(r)$ where $r = r(x,o)$. Then $\mathcal{B}(x) \geq \mathcal{B}_\gamma(x) \geq r$. Since $\mathcal{B}$ has Lipschitz constant 1 we know that $\mathcal{B}(y) \geq \frac{1}{2}r$, for all $y \in B_x(\frac{r}{2})$. By Corollary 1.1, we know that

$$v(x,t) \geq C_1 r - C_2$$

for some positive constants C_1 and C_2 independent of x. From this we conclude that $v(\cdot,t)$ grows linearly on γ. The second part of (iii) can be proved similarly. q.e.d.

Recall that M is said to satisfy (VG_k) for $k > 0$, if there exists a constant $C > 0$ such that

$$(\mathrm{VG}_k) \qquad V_o(r) \geq C r^k$$

for all $r \geq 1$. M is said to satisfy the curvature decay condition (CD) if there exists a constant $C > 0$ such that

$$(\mathrm{CD}) \qquad \fint_{B_o(r)} \mathcal{R} \leq \frac{C}{r}$$

for all $r > 0$. Finally, M is said to satisfy the fast curvature decay condition (FCD) if there is a constant $C > 0$, so that

$$(\mathrm{FCD}) \qquad \int_0^r s \left(\fint_{B_o(s)} \mathcal{R}(x)dx \right) ds \leq C \log(r+2)$$

for all $r > 0$.

Note that if the bisectional curvature is positive at some point, then $\mathcal{B}$ cannot be pluriharmonic, see Theorem 4.1 below.

Using the ideas in [7], we can prove the following.

Lemma 4.2. *Let* M^m *be a complete noncompact Kähler manifold.*

(i) *Suppose* M *supports a smooth plurisubharmonic function* u *which is strictly plurisubharmonic at* o *and suppose* u *has bounded gradient. Then* M *satisfies* (VG_m)*, where* m *is the complex dimension of* M*. If in addition,* M *has nonnegative Ricci curvature and also supports a nonconstant holomorphic function of polynomial growth then* M *satisfies* (VG_a) *for any* $a < m+1$*.*

(ii) *Suppose M has nonnegative Ricci curvature and suppose M supports a strictly plurisubharmonic function u. If $u(x) \le C(r(x)+1)$ for some constant C, then M satisfies* (CD). *If $u(x) \le C\log(r(x)+1)$ for some constant C, then M satisfies* (FCD).

Proof. (i) Let ω be the Kähler form of M which is closed. Since $\sqrt{-1}\partial\bar\partial u \ge 0$ and $\sqrt{-1}\partial\bar\partial u > 0$ at o, for any $r > 1$, there exists a smooth cutoff function $0 \le \varphi \le 1$ such that $\varphi \equiv 1$ on $B_o(r)$ and $\varphi \equiv 0$ outside $B_o(2r)$ and such that $|\nabla\varphi| \le C_1/r$ for some constant C_1 independent of r and

$$\begin{aligned}\int_{B_o(1)} \left(\sqrt{-1}\partial\bar\partial u\right)^m &\le \int_{B_o(2r)} \varphi^m \left(\sqrt{-1}\partial\bar\partial u\right)^m \\ &= -m\int_{B_o(2r)} \varphi^{m-1}\sqrt{-1}\partial\varphi\wedge\bar\partial u\wedge\left(\sqrt{-1}\partial\bar\partial u\right)^{m-1} \\ &\le \frac{mC_2}{r}\int_{B_o(2r)} \varphi^{m-1}\left(\sqrt{-1}\partial\bar\partial u\right)^{m-1}\wedge\omega\end{aligned}$$

for some constant C_2 independent of r, where we have used the fact that $|\nabla\varphi| \le C_1/r$ and $|\nabla u|$ is bounded. Continuing in this way and integrating by parts $(m-1)$ times more, we have

$$\int_{B_o(1)} \left(\sqrt{-1}\partial\bar\partial u\right)^m \le m!\cdot\left(\frac{C_2}{r}\right)^m V_o(2r).$$

Since $\partial\bar\partial u > 0$ at o, it is easy to see that M satisfies (VG_m).

If in addition, M has nonnegative Ricci curvature and supports a nonconstant polynomial growth holomorphic function f. Let $v(x) = \log(|f|^2+1)$. Then $v(x) \le C\log(r(x)+2)$, and v is plurisubharmonic. Moreover, $\partial\bar\partial v$ is not zero at every point outside a subvariety. Observe that

$$r^2 ⨍_{B_o(r)} \Delta v(y)\,dy \le C_3\log(r+2) \tag{4.1}$$

by Lemma 1.6 for some constant C_3 independent of r. On the other hand, using the cut-off function φ as above, we have that

$$
\begin{aligned}
0 &< \int_{B_o(1)} \left(\sqrt{-1}\partial\bar{\partial}v\right)\wedge\left(\sqrt{-1}\partial\bar{\partial}u\right)^{m-1} \\
&\le \int_{B_o(2r)} \varphi^m \left(\sqrt{-1}\partial\bar{\partial}v\right)\wedge\left(\sqrt{-1}\partial\bar{\partial}u\right)^{m-1} \\
&\le \frac{C_4}{r^{m-1}}\int_{B_o(2r)} \sqrt{-1}\partial\bar{\partial}v\wedge\omega^{m-1} \\
&\le \frac{C_5}{r^{m-1}}\int_{B_o(2r)} \Delta v(y)\,dy
\end{aligned}
\tag{4.2}
$$

for some constants $C_4 - C_5$ independent of r. Combining (4.1) and (4.2), we have that for some positive constant C_6 independent of r such that,

$$V_o(r) \ge C_6 \frac{r^{m+1}}{\log(r+2)}.$$

This conclues the proof of (i).

(ii) Let us prove the second statement. Our proof is basically a simplified version of [7]. Using u as a weight function, by the L^2 estimate and Theorem 3.2 of [32], there exists a nontrivial holomorphic section s of the canonical line bundle K_S (a $(n,0)$ form in terms of Theorem 3.2 of [32]) such that $s(o) \neq 0$ and

$$\int_M \|s\|^2 \exp(-C_7 u(x))\,dx = \mathcal{A} < \infty \tag{4.3}$$

for some constant $C_7 > 0$. Since $u(x) \le C\log(r(x)+2)$, for some constant C independent of x, (4.3) implies that

$$\int_{B_o(R)} \|s\|^2(x)\,dy \le (R+1)^{C_8}$$

for some constant C_8 independent of R. It is well-known that $\|s\|^2$ is subharmonic, see Lemma 4.2 of [35] for example. By the mean value inequality of Li-Schoen [25, p. 287], we have that

$$\|s\|^2(x) \le C(m) \fint_{B_o(2r(x))} \|s\|^2(y)\,dy$$

for some constant $C(m)$ depending only on m. Therefore we have that

$$\|s\|^2(x) \le (r(x)+1)^{C_9}$$

for some constant C_9 independent of x, and so

$$\log\left(||s||^2(x)+1\right) \le C_{10}\log(r(x)+2) \tag{4.4}$$

for some constant C_{10} independent of x. By Lemma 4.2 of [35] again, for any $1>\epsilon>0$, we have that

$$\Delta\log(\|s\|^2(x)+\epsilon) \ge \mathcal{R}(x)\cdot\frac{\|s\|^2}{\|s\|^2+\epsilon} \tag{4.5}$$

where $\mathcal{R}$ is the scalar curvature of M. Applying Lemma 1.6, noticing that $\mathcal{R}\cdot\frac{\|s\|^2}{\|s\|^2+\epsilon}\ge 0$, we have that

$$\begin{aligned}\int_0^r \sigma\left(\fint_{B_o(\sigma)}\mathcal{R}(x)\cdot\frac{\|s\|^2(x)}{\|s\|^2(x)+\epsilon}\,dx\right)d\sigma \\ \le C_{11}\log(r+2)-C_{12}\log(\|s\|^2(o)+\epsilon)\end{aligned} \tag{4.6}$$

for some constants C_{11} and C_{12} independent of r. Since $s(o)\ne 0$ by the construction as one can specify the value of $s(o)$ and since the set $\{s=0\}$ is of measure zero, letting $\epsilon\to 0$, the proof of the second statement in (ii) is completed.

If we only assume that u is of at most linear growth, then using similar method, instead of (4.6), we have that

$$\int_0^r \sigma\left(\fint_{B_o(\sigma)}\mathcal{R}(x)\cdot\frac{\|s\|^2(x)}{\|s\|^2(x)+\epsilon}\,dx\right)d\sigma \le C\left(r-\mathcal{R}(o)\cdot\frac{\|s\|^2(o)}{\|s\|^2(o)+\epsilon}\right)$$

for some constant C independent of r. The result follows by letting $\epsilon\to 0$ as before. q.e.d.

Our first result on the structure of complete Kähler manifolds with nonnegative bisectional curvature is a splitting theorem in terms of harmonic function and holomorphic function. Together with Lemmas 4.1 and 4.2, this theorem will be used from time to time in the rest of this section.

Theorem 4.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose f is a nonconstant harmonic function on M such that*

$$\limsup_{x\to\infty}\frac{|f(x)|}{r^{1+\epsilon}(x)}=0, \tag{4.7}$$

for any $\epsilon > 0$, where $r(x)$ is the distance of x from a fixed point. Then f must be of linear growth and M splits isometrically as $\widetilde{M} \times \mathbb{R}$. Moreover the universal cover $\overline{M}$ of M splits isometrically and holomorphically as $\widetilde{M'} \times \mathbb{C}$, where $\widetilde{M'}$ is a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that there exists a nonconstant holomorphic function f on M satisfying (4.7). Then M itself splits as $\widetilde{M} \times \mathbb{C}$.

We need the following lemmas for the proof of Theorem 4.1.

The first one is a result in [24, Corollary 5]. For the sake of completeness, we will sketch the proof. It seems that in the proof of this result, we need to assume that the holomorphic bisectional curvature is nonnegative.

Lemma 4.3. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. If f is a harmonic function with sub-quadratic growth defined on M, then f is pluriharmonic.*

Proof. Let $h = ||f_{\alpha\bar\beta}||^2 = g^{\alpha\bar\delta} g^{\gamma\bar\beta} f_{\alpha\bar\beta} f_{\gamma\bar\delta}$, where $g_{\alpha\bar\beta}$ is the metric of M and $g^{\alpha\bar\beta}$ is its inverse. Since f is harmonic, by Lemma 2.1 in [37] we know that

$$\Delta f_{\gamma\bar\delta} = -R_{\beta\bar\alpha\gamma\bar\delta} f_{\alpha\bar\beta} + \frac{1}{2}\left(R_{\gamma\bar p} f_{p\bar\delta} + R_{p\bar\delta} f_{\gamma\bar p}\right).$$

Hence in normal coordinates so that at a point x, $f_{\alpha\bar\beta} = \lambda_\alpha \delta_{\alpha\beta}$, we have

$$\begin{aligned}
\Delta h &= 2 f_{\gamma\bar\delta s\bar s} f_{\delta\bar\gamma} + ||f_{\alpha\bar\beta\gamma}||^2 + ||f_{\alpha\bar\beta\bar\gamma}||^2 \\
&= -2R_{\beta\bar\alpha\gamma\bar\delta} f_{\alpha\bar\beta} f_{\delta\bar\gamma} + \left(R_{\gamma\bar p} f_{p\bar\delta} + R_{p\bar\delta} f_{\gamma\bar p}\right) f_{\delta\bar\gamma} + ||f_{\alpha\bar\beta\gamma}||^2 + ||f_{\alpha\bar\beta\bar\gamma}||^2 \\
&= -2R_{\alpha\bar\alpha\gamma\bar\gamma}\lambda_\alpha\lambda_\gamma + 2R_{\gamma\bar\gamma}\lambda_\gamma^2 + ||f_{\alpha\bar\beta\gamma}||^2 + ||f_{\alpha\bar\beta\bar\gamma}||^2 \\
&= \sum_{\alpha,\beta} R_{\alpha\bar\alpha\beta\bar\beta}\left(\lambda_\alpha - \lambda_\beta\right)^2 + ||f_{\alpha\bar\beta\gamma}||^2 + ||f_{\alpha\bar\beta\bar\gamma}||^2 \\
&\geq 0,
\end{aligned}$$

where we have used the fact that M has nonnegative holomorphic bisectional curvature. Since $|f(x)| = o\left(r^2(x)\right)$ where $r(x)$ is the distance from a fixed point $o \in M$, as in [24, pp. 90-91], we have

$$\frac{1}{V_o(R)} \int_{B_o(R)} h \leq \frac{C}{R^{-2} V_o(R)} \int_{B_o(R)} |\nabla f|^2 = o(1),$$

as $R \to \infty$. Here C is a constant independent of R and we has used the gradient estimate in [8]. Since h is subharmonic, $h \equiv 0$ by the mean value inequality in [25]. Hence f is pluriharmonic. q.e.d.

Lemma 4.4. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let f be a pluriharmonic function. Then $\log(1+|\nabla f|^2)$ is plurisubharmonic.*

Proof. We adapt the complex notation. Let

$$h = |\nabla f|^2 = \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} f_\alpha f_{\bar{\beta}}.$$

Here $g_{\alpha\bar{\beta}}$ is the Kähler metric and $(g^{\alpha\bar{\beta}})$ is the inverse of $(g_{\alpha\bar{\beta}})$. To prove that $\log(1+h)$ is plurisubharmonic, it is sufficient to show that $[\log(1+h)]_{\gamma\bar{\gamma}} \geq 0$ in normal coordinates. Direct calculation shows that:

$$\begin{aligned}
h_{\gamma\bar{\gamma}} &= \left(\sum_{\alpha\beta} g^{\alpha\bar{\beta}} f_\alpha f_{\bar{\beta}}\right)_{\gamma\bar{\gamma}} \\
&= \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} \left[f_{\alpha\gamma} f_{\bar{\beta}\bar{\gamma}} + f_{\alpha\bar{\gamma}} f_{\bar{\beta}\gamma} + f_{\alpha\gamma\bar{\gamma}} f_{\bar{\beta}} + f_\alpha f_{\bar{\beta}\gamma\bar{\gamma}}\right] \\
&= \sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha,s} R_{\gamma\bar{\gamma}\alpha\bar{s}} f_s f_{\bar{\alpha}}
\end{aligned} \tag{4.8}$$

where we have used the fact that f is pluriharmonic. Hence

(4.9)

$$\begin{aligned}
[\log(1+h)]_{\gamma\bar{\gamma}} &= \frac{1}{(1+h)^2}\left[(1+h)h_{\gamma\bar{\gamma}} - h_\gamma h_{\bar{\gamma}}\right] \\
&= \frac{1}{(1+h)^2}\Bigg[(1+h)\left(\sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha,s} R_{\gamma\bar{\gamma}\alpha\bar{s}} f_s f_{\bar{\alpha}}\right) \\
&\qquad - \sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}} \sum_{\alpha} f_\alpha f_{\bar{\alpha}\bar{\gamma}}\Bigg] \\
&\geq \frac{1}{(1+h)^2}\left(\sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha,s} R_{\gamma\bar{\gamma}\alpha\bar{s}} f_s f_{\bar{\alpha}}\right)
\end{aligned}$$

where we have used the fact that f is pluriharmonic. From (4.9), the fact that M has nonnegative holomorphic bisectional curvature, it is easy to see that $\log(1+h)$ is plurisubharmonic. q.e.d.

Proof of Theorem 4.1. Let f be a nonconstant harmonic function on M satisfying (4.7). Then f is pluriharmonic by Lemma 4.3. By Lemma 4.4, the function $u = \log(1 + |\nabla f|^2)$ is plurisubharmonic. By the gradient estimates in [8], $|u|(x) = o(\log r(x))$. By Theorem 3.2, we conclude that $|\nabla f|$ is constant. Hence f must be of linear growth. Moreover, by the Bochner formula, we conclude that ∇f must be parallel. Hence $J(\nabla f)$ is also parallel, where J is the complex structure of M. From this it is easy to see that the universal cover of M splits as $\widetilde{M}' \times \mathbb{C}$ isometrically and holomorphically. At the same time by integrating along ∇f, M splits as $\widetilde{M} \times \mathbb{R}$ isometrically, where $\widetilde{M}$ can be taken to be the component of $f^{-1}(0)$. In the case that M supports a nonconstant holomorphic function of growth rate (4.7), both the real and imaginary part will split a factor of $\mathbb{R}$ and clearly that they consist a complex plane $\mathbb{C}$. q.e.d.

An easy consequence is that if the Ricci curvature is positive at some point of the manifold then any harmonic function satisfying (4.7) must be a constant.

In the next theorem, we shall give some results on the structure of complete noncompact Kähler manifold with nonnegative sectional or holomorphic bisectional curvature.

Theorem 4.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature.*

(i) *Suppose M is simply connected, then $M = N \times M'$ holomorphically and isometrically, where N is a compact simply connected Kähler manifold, M' is a complete noncompact Kähler manifold and both N and M' have nonnegative holomorphic bisectional curvature. Moreover, M' supports a smooth strictly plurisubharmonic function with bounded gradient and satisfies* (VG_k) *and* (CD)*, where k is the complex dimension of M'. If, in addition, M has nonnegative sectional curvature outside a compact set, then M' is also Stein.*

(ii) *If the holomorphic bisectional curvature of M is positive at some point, then M itself supports a smooth strictly plurisubharmonic function with bounded gradient, and satisfies* (VG_m) *and* (CD)*, where m is the complex dimension of M. If, in addition, M has nonnegative sectional curvature outside a compact set, then M is also Stein.*

Remark 4.1.

(a) The factor N in (i) may not be present. In this case, $M = M'$ and satisfies the conditions on M' mentioned in the theorem. This kind of remark also applies to Theorem 4.3.

(b) It was first proved in [7] that M satisfies (VG_m) if the holomorphic bisectional curvature nonnegative and is positive at some point, and that M satisfies (CD) if M has positive holomorphic bisectional curvature everywhere.

(c) By [29, 19] (see also [2]), N in (i) is a compact Hermitian symmetric manifold.

Proof. Let $\mathcal{B}$ be the Busemann function of M and let v be the solution of the heat equation with initial value $\mathcal{B}$. Let T_0 be as in Lemma 4.1.

(i) Let $0 < t < T_0$ be fixed and let $u(x) = v(x,t)$. By Lemma 4.1, suppose M is simply connected, then $M = N_1 \times M_1$ isometrically and holomorphically so that $u_{\alpha\bar\beta} \equiv 0$ when restricted on N_1 and $u_{\alpha\bar\beta} > 0$ when restricted to M_1. Suppose N_1 is not compact, then there is a ray of M which lies on N_1. By Lemma 4.1(iii), u is not constant on N_1. However, u has bounded gradient by Lemma 4.1(ii). Theorem 4.1 then implies that $N_1 = N_2 \times \mathbb{C}$ isometrically and holomorphically. Continuing in this way, we conclude that $N_1 = N \times \mathbb{C}^\ell$ isometrically holomorphically for some $\ell \geq 0$, where N is a compact simply connected with nonnegative holomorphic bisectional curvature. Let $M' = \mathbb{C}^\ell \times M_1$. Then M' supports a strictly plurisubharmonic function with bounded gradient and hence also satisfies (VG_k) and (CD) by Lemma 4.2, where $k = \dim_{\mathbb{C}} M'$.

If, in addition, M has nonnegative sectional curvature outside a compact set, then $\mathcal{B}$ is an exhaustion function by [5, 12]. Hence u is an exhaustion function by Lemma 4.2. Therefore M' in the above is also Stein.

(ii) Suppose the holomorphic bisectional curvature of M is positive at some point. Then $\mathcal{B}$ cannot be pluriharmonic. Otherwise, since $\mathcal{B}$ is of linear growth and nonconstant, there will be a factor of $\mathbb{C}$ splitted from the universal cover of M by Theorem 4.1. Hence u is strictly plurisubharmonic by Lemma 4.1(i). The rest of the proof is similar to (i). q.e.d.

In [45], Wu proved that a complete noncompact Kähler manifold is Stein if it has nonnegative sectional curvature outside a compact set, with nonnegative holomorphic bisectional curvature everywhere which is positive outside a compact set. The last statement of Theorem 4.2(ii) is a generalization of this result.

In the last part of Theorem 4.2(i) or (ii), the assumption on the sectional curvature is needed only for the proof that the Busemann function is an exhaustion function. In some cases, this is true even if we only assume that the Ricci curvature is nonnegative. Hence we have the following result.

Corollary 4.1. *Let M^m be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that M is of maximum volume growth or M has a pole. Then M is Stein. Moreover, M satisfies* (CD)*.*

Proof. Suppose M has maximum volume growth. Let $\widetilde{M}$ be the universal cover of M, then $\widetilde{M}$ has maximum volume growth and $\pi : \widetilde{M} \to M$ is a finite cover by [23, p. 10]. Suppose $\widetilde{M}$ is Stein, then $\widetilde{M}$ has a smooth strictly plurisubharmonic exhaustion function f. Then the function $h(x) = \sum_{\widetilde{x}} f(\widetilde{x})$ for $x \in M$, where the summation is taken over all $\widetilde{x} \in \widetilde{M}$ so that $\pi(\widetilde{x}) = x$. Then h is a strictly plurisubharmonic exhaustion function of M. Hence M is also Stein. So without loss of generality, we may assume that M is simply connected.

Since M has maximum volume growth, the Busemann function $\mathcal{B}$ is an exhaustion function by [38, pp. 400-401]. Let u be as in the proof of Theorem 4.2, then by this theorem, $M = N \times M'$ as described in the theorem. Since M has maximum volume growth, the factor N will not be present. Hence $M = \mathbb{C}^\ell \times M_1$ holomorphically and isometrically, so that u is strictly plurisubharmonic on M_1. By Lemma 4.1, it is also an exhaustion function on M_1. Therefore M_1 must be Stein by [10] and so M is also Stein. The last statement follows from Lemma 4.2.

Suppose M has a pole, then it is easy to see that the Busemann function with respect to the pole is an exhaustion function. The manifold is diffeomorphic to $\mathbb{R}^{2m}$. One can conclude that the splitting given by Theorem 4.2 contains no compact factor. One can then proceed as above to conclude that M is Stein. q.e.d.

In [46], it was proved that M is Stein under the assumption that M has a pole and nonnegative bisectional curvature which is positive outside a compact subset of M. Our result answers affirmatively the question raised in [46, page 255] for the nonnegative bisectional curvature case. Under the maximum volume growth assumption, if the holomorphic bisectional curvature is actually positive everywhere then it is easy to see that it is Stein by the results on smooth approximation of strictly plurisubharmonic function in [15] and the result in [38] mentioned above. This was observed in [44]. Under the maximum volume growth and the nonnegativity of the bisectional curvature assumptions together with the additional assumption that the curvature decays like r^{-1-a}, the result was proved in [6]. This kind of results are related to a conjecture by Greene-Wu [14] and Siu [40] that a complete noncompact Kähler manifold with positive bisectional curvature is Stein.

Without assuming that M is simply connected or the holomorphic bisectional curvature of M is positive at some point, by applying Theorem 4.1 inductively, we immediately have:

Corollary 4.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Then M have the holomorphic-isometric splitting $M^m = \mathbb{C}^k \times M_2^{m-k}$. Here M_2 is a complete Kähler manifold of nonnegative bisectional curvature with the property that any holomorphic function on M_2 satisfying (4.7) must be a constant.*

There is an open question whether the ring of polynomial growth holomorphic functions on a complete noncompact Kähler manifold with nonnegative curvature is finitely generated, see [52, p. 391, Problem 63]. This motivates us to study the factor M' in Theorem 4.2(i) or M in Theorem 4.2(ii) in more details. We have the following further splitting.

Theorem 4.3. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Assume that M supports a smooth strictly plurisubharmonic function u on M with bounded gradient.*

(i) *If M is simply connected, then $M = \mathbb{C}^\ell \times M_1 \times M_2$ isometrically and holomorphically for some $\ell \geq 0$, where M_1 and M_2 are complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that any polynomial growth holomorphic function on M is independent of the factor M_2, and any linear growth holomorphic function is independent of the factor M_1 and M_2. Moreover, M_1 supports a strictly plurisubharmonic function of logarithmic growth and satisfies* (FCD) *and* (VG$_a$)*, for* *any $a < k+1$, where $k = \dim_{\mathbb{C}} M_1$.*

(ii) *Suppose the holomorphic bisectional curvature of M is positive at some point, then either M has no nonconstant polynomial growth holomorphic function or M itself satisfies* (FCD) *and* (VG_a)*, for any $a < m+1$.*

Proof. (i) We prove this part of the theorem by induction on the dimension of M^m.

Suppose M does not support any nontrivial polynomial growth holomorphic function, then we simply take $M_2 = M$ and the factors $\mathbb{C}^\ell$ and M_1 are not present. Suppose there is a nontrivial polynomial growth holomorphic function f on M. Let $w = \log(1 + |f|^2)$. It is easy to see that w is a plurisubharmonic function so that

$$0 \le w(x) \le C_1 \log(r(x) + 2) \tag{4.10}$$

We can solve the Cauchy problem (1.6) with initial data $w(x)$. Denote $\widetilde{w}(x,t)$ to be the solution. Note that $\widetilde{w}(\cdot,t)$ is nonconstant because w is nonconstant. We can apply Theorem 3.1 again to conclude that there exists $t > 0$ and a parallel distribution $\mathcal{K}$ which corresponding to the null space of $\widetilde{w}_{\alpha\bar{\beta}}(x,t)$. Suppose $\dim \mathcal{K} = 0$, then $\widetilde{w}(\cdot,t)$ is strictly plurisubharmonic and with logarithmic growth by Corollary 1.1. Then $M = \mathbb{C}^\ell \times M_1$ by Corollary 4.2 so that every linear growth holomorphic function on M is independent of M_1. M_2 is not present in this case. Moreover, M_1 satisfies (FCD) and (VG_a) by Lemma 4.2.

Suppose $\dim \mathcal{K} > 0$, then $M = N_1 \times N_2$, such that $\widetilde{w}_{\alpha\bar{\beta}}(\cdot,t) \equiv 0$ when restricted on N_1, $\widetilde{w}_{\alpha\bar{\beta}}(\cdot,t) > 0$ when restricted on N_2. They are simply connected, complete Kähler manifolds with nonnegative holomorphic bisectional curvature. $\dim_{\mathbb{C}} N_1 = \dim \mathcal{K} > 0$, but $\dim_{\mathbb{C}} N_1 < \dim_{\mathbb{C}} M$. Otherwise, $\widetilde{w}(\cdot,t)$ is harmonic on M and is constant by (4.8) and [8]. Hence the dimensions of N_1 and N_2 are both less than m. They are also noncompact because M supports a strictly plurisubharmonic function. Hence N_1 and N_2 are Kähler manifolds satisfy the same conditions satisfied by M. By induction hypothesis $N_1 = \mathbb{C}^{\ell_1} \times N_{1,1} \times N_{1,2}$ and $N_2 = \mathbb{C}^{\ell_2} \times N_{2,1} \times N_{2,2}$ isometrically holomorphically, such that for $j = 1,\ 2$, every polynomial growth holomorphic function on N_j is independent of the factor $N_{j,2}$ and every linear growth holomorphic function is independent of the factor $N_{j,1} \times N_{j,2}$. $N_{j,1}$ satisfies (FCD) and (VG_{k_j}) where $k_j = \dim_{\mathbb{C}} N_{j,1}$. Let $M_1 = N_{1,1} \times N_{2,1}$, $M_2 = N_{1,2} \times N_{2,2}$ and $\ell = \ell_1 + \ell_2$. Then $M = \mathbb{C}^\ell \times M_1 \times M_2$ isometrically holomorphically.

Since every polynomial (respectively linear) growth holomorphic function on M is still a polynomial (respectively linear) growth holomorphic function when restricted on N_1 and N_2, hence the splitting satisfies all the required conditions if we can prove that M_1 also satisfies the required volume growth and curvature decay conditions.

The volume growth condition is satisfied by M_1 because of the corresponding volume growth conditions are satisfied by $N_{1,1}$ and $N_{2,1}$. Moreover, for $r > 0$, if $\mathcal{R}_1$, $\mathcal{R}'$ and $\mathcal{R}''$ are the scalar curvatures of M_1, $N_{1,1}$ and $N_{2,1}$ respectively, and if $B_{o_1}(s)$, $B_{o'}(s)$ and $B_{o''}(s)$ are the geodesic balls of M_1, $N_{1,1}$ and $N_{2,1}$ respectively where $o_1 = (o', o'')$, then

$$\fint_{B_{o_1}(s)} \mathcal{R}_1 \le C \left(\fint_{B_{o'}(s)} \mathcal{R}' + \fint_{B_{o''}(s)} \mathcal{R}'' \right)$$

for some constant C depending only on the dimensions of M_1, $N_{1,1}$ and $N_{2,1}$. Since $N_{j,1}$ satisfies (FCD) for $j = 1,\ 2$, M_1 also satisfies (FCD).

From the above proof, it is easy to see that part (i) is true if $m = 1$. The proof of Theorem 4.3(i) is then completed.

(ii) If the holomorphic bisectional curvature of M is positive at some point, suppose M supports no nonconstant polynomial growth holomorphic function, then we are done. Otherwise, let f be a nontrivial polynomial growth holomorphic function. Construct w and $\widetilde{w}$ as in the proof of (i), then there exists $t > 0$, $\widetilde{w}(\cdot, t)$ must be strictly plurisubharmonic by Theorem 3.1(iii). One can proceed as in the proof of (i). q.e.d.

Remark 4.2. By the Theorems 4.2 and 4.3, in order to study polynomial growth holomorphic functions on a complete noncompact Kähler M^m with nonnegative holomorphic bisectional curvature which is either simply connected or has positive holomorphic bisectional curvature at some point, we may assume that M satisfies the curvature decay condition (FCD) and the volume growth condition (VG_a) for any $a < m+1$. Note that under a very mild condition on the bound of the scalar curvature, if it decays faster than (FCD), then manifold must be flat. We shall discuss this problem in a later section.

As a simple consequence of Theorems 4.2 and 4.3 we have the following uniformization type result.

Corollary 4.3. *Let M be a complete, simply-connected, Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that the volume growth of M satisfies $V_o(r) = o(r^2)$. Then M is biholo-*

morphic to $N \times \mathbb{C}$, where N is biholomorphic to a compact Hermitian manifold. If M supports nonconstant holomorphic functions of polynomial growth the same result holds if $V_o(r) = O(r^a)$, for some $a < 3$.

5. Structure of nonnegatively curved Kähler manifolds II

In [43], Takayama proved that if M is a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and negative canonical bundle and if M supports a continuous plurisubharmonic exhaustion function, then M has a structure of holomorphic fibre bundle over a Stein manifold whose fibre is biholomorphic to some compact Hermitian symmetric manifold. In particular, the result applies to Kähler manifolds with nonnegative sectional curvature and positive Ricci curvature. This settled a conjecture of Greene-Wu [14, §3] that a complete noncompact Kähler manifold with nonnegative sectional curvature and positive Ricci curvature is holomorphically convex. In this section, we shall give more detailed results on the structure on the class of manifolds related to the above conjecture. We shall also include results of Fangyang Zheng [54] on the structure of complete noncompact Kähler manifold with nonnegative sectional curvature. The authors are grateful to Fangyang Zheng for allowing them to include his results and proofs in this work.

Before we state our results, let us make some preparations. Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let $\mathcal{B}$ be the Busemann function at a point $o \in M$, and let v be the solution to the heat equation with initial value $\mathcal{B}$. Then there is $t_0 > 0$ such that $v(x, t_0)$ is smooth plurisubharmonic and the kernel of $\mathcal{K}(x, t_0)$ of $v_{\alpha\bar{\beta}}(x, t_0)$ is a smooth, parallel distribution on M. In the following, we shall suppress the variable t_0 and just write $v(x) = v(x, t_0)$. Note that if $\mathcal{B}$ is an exhaustion function of M, then $v(x)$ is also an exhaustion function. Moreover, v has bounded gradient. All these results are contained in Lemma 4.1.

Let $\widetilde{M}$ be the universal cover of M with projection $\widetilde{\pi}$ and let $\widetilde{v} = v \circ \widetilde{\pi}$. Then $\widetilde{M} = \widetilde{N} \times \widetilde{L}$ isometrically and holomorphically. In the following, a point in $\widetilde{M}$ will be denoted by (y, z) etc. The splitting of $\widetilde{M}$ has the following properties. For each $z \in \widetilde{L}$, $v_{\alpha\bar{\beta}} \equiv 0$ on $\widetilde{N}_z = \widetilde{N} \times \{z\}$ and for each $y \in \widetilde{N}$, $v_{\alpha\bar{\beta}} > 0$ when restricted on $\widetilde{L}_y = \{y\} \times \widetilde{L}$. That is, $\widetilde{N}_z$, for $z \in \widetilde{L}$ are integral manifolds of the distribution $\left(\widetilde{\pi}^{-1}\right)_* (\mathcal{K})$.

Let Γ be the fundamental group of M.

Lemma 5.1. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that the Busemann function $\mathcal{B}$ of M is an exhaustion function and suppose the universal cover $\widetilde{M}$ contains no Euclidean factor. Then with the above notations, $\widetilde{N}$ is compact and a deck transformation $\gamma \in \Gamma$ is of the form $\gamma(y,z) = (\gamma_1(y), \gamma_2(z))$ so that γ_1 and γ_2 are holomorphic isometries of $\widetilde{N}$ and $\widetilde{L}$ respectively. Moreover γ_2 has no fixed point unless γ is the identity.*

Proof. Let us first prove that $\widetilde{N}$ is compact. Fix $z \in \widetilde{L}$ and consider $\widetilde{N}_z = \widetilde{N} \times \{z\}$. By the construction of $\widetilde{N}$, $\widetilde{v}$ is pluriharmonic on $\widetilde{N}_z$. Suppose $\widetilde{v}$ is not constant on $\widetilde{N}_z$, then $\widetilde{N}_z$ will contain a factor $\mathbb{C}$ by Theorem 4.1. This contradicts the assumption that $\widetilde{M}$ does not contain any Euclidean factor. Hence $\widetilde{v}$ must be constant on $\widetilde{N}_z$. Since $\widetilde{v}$ is the lift of v and since $\mathcal{B}$ and hence v is an exhaustion function, we conclude that $\widetilde{\pi}(\widetilde{N}_z)$ is a bounded and hence its closure K in M is compact. Since $\widetilde{\pi}$ is a covering map, there exists a compact set $\widetilde{K}$ in $\widetilde{M}$ such that $\widetilde{\pi}(\widetilde{K}) \supset K$. We can now proceed as in [4, p. 126]. Suppose $\widetilde{N}_z$ is not compact, then there is a ray σ in $\widetilde{N}_z$. Since $\widetilde{\pi}(\sigma(n)) \in K$, there exists $\gamma_n \in \Gamma$ such that $\gamma_n(\sigma(n)) \in \widetilde{K}$. Since $\widetilde{K}$ is compact, passing to a subsequence if necessary, we may assume that $\gamma_n(\sigma(n)) \to p \in \widetilde{M}$ and $(\gamma_n)_*(\sigma'(n)) \to \vec{w} \in T_p(\widetilde{M})$. Let τ be the geodesic with $\tau(0) = p$ and $\tau'(0) = \vec{w}$, then it is easy to see that τ is a line. By [4], $\widetilde{M}$ has a factor of $\mathbb{R}$. This is a contradiction. Hence $\widetilde{N}$ is compact.

Let $\gamma \in \Gamma$, then $\mathrm{Proj}_2\left(\gamma(\widetilde{N}_z)\right)$ is a compact subvariety in $\widetilde{L}$ where Proj_2 is the projection onto $\widetilde{L}$. Since $\widetilde{L}$ supports a strictly plurisubharmonic function, it must be a point. Hence γ is of the form $\gamma(y,z) = (f(y,z), g(z))$. Since γ is an isometry, g will not increase length. This is also true for g^{-1}, and hence it is easy to see that g is a local isometry and $f(y,z) = f(y)$. Therefore $\gamma(y,z) = (\gamma_1(y), \gamma_2(z))$, where γ_1 and γ_2 are holomorphic isometries on $\widetilde{N}$ and $\widetilde{L}$ respectively. Suppose γ is not the identity and suppose $\gamma_2(z) = z$ for some $z \in \widetilde{L}$, then γ_1 will not have any fixed point. Then $\widetilde{N}$ will cover a compact complex manifold with nonnegative holomorphic bisectional curvature. This is impossible by [19]. This completes the proof of the lemma. q.e.d.

Let Γ_2 be the subgroup of the isometry group of $\widetilde{L}$ which is the image under the map $\gamma \to \gamma_2$. By Lemma 5.1 Γ_2 acts freely and holomorphically on $\widetilde{L}$. Let $\widehat{M} = \widetilde{L}/\Gamma_2$. Since $M = \left(\widetilde{N} \times \widetilde{L}\right)/\Gamma$, by Lemma 5.1

there is a projection $\pi_r : M \to \widehat{M}$ such that the following diagram commutes:

$$\begin{array}{ccc}\widetilde{N} \times \widetilde{L} & \xrightarrow{\text{Proj}_2} & \widetilde{L} \\ {\scriptstyle\widetilde{\pi}}\Big\downarrow & & {\scriptstyle\widehat{\pi}}\Big\downarrow \\ M & \xrightarrow{\pi_r} & \widehat{M}.\end{array}$$

In fact, $\pi_r(\widetilde{\pi}(y,z)) = \widehat{\pi}(z)$.

Theorem 5.1. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that the Busemann function is an exhaustion function. Suppose the universal cover $\widetilde{M}$ has no Euclidean factor. Using the above notations, we have the following:*

- (i) $\widetilde{M} = \widetilde{N} \times \widetilde{L}$ *and* $\widetilde{N}$ *is compact.*
- (ii) $\pi_r : M \to \widehat{M}$ *has the structure of a holomorphic fibre bundle, where each fibre is isometrically biholomorphic to N.*
- (iii) $\widehat{\pi} : \widetilde{L} \to \widehat{M}$ *is a holomorphic and Riemannian covering map. $\widehat{M}$ is a complete noncomapct Kähler and Stein manifold with nonnegative holomorphic bisectional curvature. $\widetilde{L}$ is also Stein.*

Proof. (i) follows from Lemma 5.1.

To prove (ii), let $z \in \widetilde{L}$ and let $\widehat{U}$ be a neighborhood of $\widehat{\pi}(z)$ in $\widehat{M}$ which is evenly covered by a family $\mathcal{F}$ of neighborhoods in $\widetilde{L}$. Let $\widetilde{W_1}$ and $\widetilde{W_2}$ be two members in $\mathcal{F}$, then there exists $\gamma_2 \in \Gamma_2$ such that $\gamma_2(\widetilde{W_1}) = \widetilde{W_2}$. Suppose γ_2 is such that $\gamma = (\gamma_1, \gamma_2)$ for some $\gamma \in \Gamma$. Then $\gamma(\widetilde{N} \times \widetilde{W_1}) = \widetilde{N} \times \widetilde{W_2}$ and $\widetilde{\pi}$ maps $\widetilde{N} \times \widetilde{W_1}$ holomorphically and isometrically onto its image. Let $\widetilde{W_1}$ be one of the neighborhoods in $\mathcal{F}$. It is easy to see that $\pi_r^{-1}(\widehat{U}) = \widetilde{\pi}(\bigcup_{\widetilde{W} \in \mathcal{F}} \widetilde{N} \times \widetilde{W}) = \widetilde{\pi}(\widetilde{N} \times \widetilde{W_1})$ which is holomorphically isometric to $\widetilde{N} \times \widetilde{W_1}$. Each fibre $\pi_r^{-1}(\widehat{\pi}(z)) = \widetilde{\pi}(\widetilde{N}_z)$ which is holomorphically isometric to $\widetilde{N}$. This completes the proof of (ii).

(iii) Let $\widetilde{v}$ be the smooth plurisubharmonic function on $\widetilde{M}$ defined before Lemma 5.1. Since $\widetilde{N}$ is compact, $\widetilde{v}(y,z) = \widetilde{v}(z)$ which is independent of the factor $\widetilde{N}$. Since $\widetilde{v} = v \circ \widetilde{\pi}$, for any $\gamma \in \Gamma$, $\widetilde{v}(y,z) = \widetilde{v}(\gamma(y,z))$. Hence $\widetilde{v}$ is equivariant with respect to Γ_2 and so it descends to be a smooth strictly plurisubharmonic function $\hat{v}$ on $\widehat{M}$ because $\widetilde{v}$ is strictly plurisubharmonic on $\widetilde{L}$. Note also that $v(x) = \hat{v}(\pi_r(x))$. Since v is an exhaustion function on M, $\hat{v}$ is an exhaustion function of $\widehat{M}$. Hence $\widehat{M}$ is Stein by [10]. The fact that $\widetilde{L}$ is Stein follows from a result of [41] that the universal cover of a Stein manifold is Stein. q.e.d.

Remark 5.1. The condition in the theorem that $\mathcal{B}$ is an exhaustion function will be satisfied if M has nonnegative sectional curvature outside a compact set, see [5, 12]. The condition that $\widetilde{M}$ has no Euclidean factor will be satisfied if the Ricci curvature of M is positive at some point. Hence if M has nonnegative sectional curvature outside a compact set (in addition to the fact that M has nonnegative holomorphic bisectional curvature) and if M has positive Ricci curvature is at some point, then M has the structure as described in the theorem.

The example of Cousin [9] (see also [31, page 839], [43, page 141]) shows that the structure results in Theorem 5.1(ii) and (iii) are not true if there is a Euclidean factor in $\widetilde{M}$. In this case, M might still be a fibre bundle but the base space may not be Stein.

Suppose M has nonnegative sectional curvature everywhere, then Fangyang Zheng [54] obtains the following stronger structure theorem.

Theorem 5.2 (Zheng)**.** *Let M be a complete noncompact Kähler manifold with nonnegative sectional curvature. Assume that the universal cover $\widetilde{M}$ of M does not have Euclidean factors. Then M is simply connected and $M = N \times L$ isometrically and holomorphically where N is a compact Hermitian symmetric manifold and L is diffeomorphic to the Euclidean space $\mathbb{R}^{2\ell}$ where $\ell = \dim_{\mathbb{C}} L$.*

The theorem follows immediately from the following lemma and Theorem 5.1.

Lemma 5.2. *Let M^m be a complete noncompact Kähler manifold with nonnegative sectional curvature so that its universal cover does not contain a Euclidean factor. Suppose that M supports a strictly plurisubharmonic function. Then the soul of M is a point and hence M is simply connected and is diffeomorphic to $\mathbb{R}^{2m}$.*

Let us assume the lemma is true for the moment.

Proof of Theorem 5.2. Apply Theorem 5.1 to M. Using the notations as in Theorem 5.1, the manifold $\widehat{M}$ satisfies all the conditions in the lemma because $\widetilde{L}$ contains no Euclidean factor. Hence $\widehat{M}$ is simply connected. So M is also simply connected because it is a fibre bundle

over $\widehat{M}$ with fibre N which is simply connected. Hence $M = \widetilde{M}$. By the lemma again, we know that $\widetilde{L}$ is diffeomorphic to the Euclidean space.
q.e.d.

It remains to prove Lemma 5.2.

Proof of Lemma 5.2. Let $\widetilde{M}$ be the universal cover of M with covering map π and let $\mathcal{S}$ be a soul of M. Let $\widetilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$. Since $\mathcal{S}$ is totally geodesic and totally convex by [5], $\widetilde{\mathcal{S}}$ is also totally geodesic and totally convex. In particular, $\widetilde{\mathcal{S}}$ is connected. Suppose $\widetilde{\mathcal{S}}$ is noncompact, then it contains a ray which will also be a ray in $\widetilde{M}$. Since $\pi(\widetilde{\mathcal{S}}) = \mathcal{S}$ which is compact, we can conclude as in the proof of Lemma 5.1 that $\widetilde{\mathcal{S}}$ is compact because $\widetilde{M}$ contains no Euclidean factor. Moreover, $\widetilde{\mathcal{S}}$ is simply connected by [5, Theorem 2.1]. We want to prove that $\widetilde{\mathcal{S}}$ is a point. It will be sufficient to prove that $\widetilde{\mathcal{S}}$ is flat because $\widetilde{\mathcal{S}}$ is connected, simply connected and compact.

By [5, Theorem 3.1], since $\mathcal{S}$ is a soul of M and $\widetilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$, we have

$$R(u, v)v = R(v, u)u = 0 \tag{5.1}$$

for any point $p \in \widetilde{\mathcal{S}}$, any vector $u \in T_p(\widetilde{\mathcal{S}})$ and v which is normal to $T_p(\widetilde{\mathcal{S}})$. Here R is the Riemannian curvature tensor of $\widetilde{M}$. Let J be the complex structure of $\widetilde{M}$. Let $p \in \widetilde{\mathcal{S}}$ and let W be the subspace of $V = T_p(\widetilde{\mathcal{S}})$ consisting of vectors $v \in V$ such that $Jv \in V$. Let γ be a piecewise smooth closed curve on $\widetilde{\mathcal{S}}$ from p. Since $\widetilde{\mathcal{S}}$ is totally geodesic, J commutes with parallel translation along γ on $\widetilde{\mathcal{S}}$. Hence W is invariant under parallel translation along γ. Note also that $J(W) = W$. Since $\widetilde{\mathcal{S}}$ is simply connected, $\widetilde{\mathcal{S}}$ can be splitted as a product with a factor $\widetilde{\mathcal{S}}_1$ whose tangent spaces are invariant under J. By assumption, M and hence $\widetilde{M}$ supports a strictly plurisubharmonic function. This implies that $\widetilde{\mathcal{S}}_1$ is a point and $W = \{0\}$. That is to say

$$JV \cap V = \{0\}. \tag{5.2}$$

For $v \in V$, let $A(v)$ be the orthogonal projection of Jv onto V. Since $J + J^t = 0$, it is easy to see that $A + A^t = 0$. We can find orthonormal basis $e_1, e_2, \dots, e_{2k-1}, e_{2k}, e_{2k+1}, \dots, e_s$, $s = \dim_{\mathbb{R}} V$, under which A takes the block diagonal form $A = \mathrm{diag}\{\delta_1 J_2, \cdots, \delta_k J_2, 0_l\}$, where $k \geq 0$, $1 \geq \delta_1 \geq \cdots \geq \delta_k > 0$, $l = \dim(S) - 2k$,

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and 0_l is a zero matrix. By (5.2), we have $\delta_j < 1$.

Now take $e_1,\ e_2$ for example. We have

$$Je_1 = \alpha e_2 + v_1, \quad Je_2 = -\alpha e_1 + v_2,$$

where $v_1,\ v_2 \in V^\perp$. Since $0 < \alpha = \delta_1 < 1$, v_1, v_2 are nonzero vectors. Since $\langle Je_i, Je_j\rangle = \delta_{ij}$, we conclude that $\langle v_i, v_j\rangle = \beta^2\delta_{ij}$, where $\beta = ||v_1|| = ||v_2|| = \sqrt{1-\alpha^2}$. Let $v_3 = \beta^{-1}v_1$ and let $v_4 = \beta^{-1}v_2$. Then we have

$$Je_1 = \alpha e_2 + \beta v_3, \quad Je_2 = -\alpha e_1 + \beta v_4. \tag{5.3}$$

Using the fact that $J^2 = -I$, from (5.3), we also have

$$Je_3 = -\beta e_1 - \alpha v_4, \quad Je_4 = -\beta e_2 + \alpha v_3. \tag{5.4}$$

By (5.1) and (5.3), since M is Kähler, we have

$$\begin{aligned} 0 &= -R(e_1, v_3, Jv_4, e_1) = R(e_1, v_3, v_4, Je_1) \\ &= R(e_1, v_3, v_4, \alpha e_2 + \beta v_3) = \alpha R(e_1, v_3, v_4, e_2). \end{aligned}$$

By (5.1) and (5.3),

$$\begin{aligned} 0 &= -R(e_1, v_4, Je_2, e_1) = R(e_1, v_4, e_2, Je_1) \\ &= R(e_1, v_4, e_2, \alpha e_2 + \beta v_3) = \beta R(e_1, v_4, e_2, v_3). \end{aligned}$$

Hence by the Bianchi identity, we have

$$R(e_1, e_2, v_3, v_4) = -R(e_1, v_3, v_4, e_2) - R(e_1, v_4, e_2, v_3) = 0.$$

So

$$\begin{aligned} 0 &= R(e_1, e_2, v_3, v_4) \\ &= R(e_1, e_2, Jv_3, Jv_4) \\ &= R(e_1, e_2, -\beta e_1 - \alpha v_4, -\beta e_2 + \alpha v_3) \\ &= R(e_1, e_2, -\beta e_1, -\beta e_2) + R(e_1, e_2, -\beta e_1, \alpha v_3) \\ &\quad + R(e_1, e_2, -\alpha v_4, -\beta e_2) + R(e_1, e_2, -\alpha v_4, \alpha v_3) \\ &= \beta^2 R(e_1, e_2, e_1, e_2) \end{aligned} \tag{5.5}$$

where we have used (5.1)and (5.4). Using (5.1) and (5.4), we also have

$$\begin{aligned} R(e_1, Je_1, e_1, Je_1) &= R(e_1, \alpha e_2 + \beta v_3, e_1, \alpha e_2 + \beta v_3) \\ &= \alpha^2 R(e_1, e_2, e_1, e_2) = 0. \end{aligned} \tag{5.6}$$

Now we use the following fact in [53]. Suppose X is such that the holomorphic sectional curvature $R(X, \bar{X}, X, \bar{X}) = 0$ and if the sectional curvature is also nonnegative, then $R(X, \bar{X}, Y, \bar{Y}) = 0$ for any Y. Hence (5.6) implies that the sectional curvature $K(e_1, u)$ of the plane spanned by e_1 and any tangent vector $u \in T_p(\widetilde{\mathcal{S}})$ is zero. Similarly, we can prove that $K(e_j, u) = 0$ for $1 \leq j \leq 2k$. Since $Je_j \in V^\perp$ for $2k+1 \leq j \leq s$, $K(e_j, Je_j) = 0$ by (5.1). Hence we have $K(e_j, u) = 0$ for all j and for all $u \in T_p(\widetilde{\mathcal{S}})$. Since p is any point on $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{S}}$ is flat. This completes the proof of the lemma. q.e.d.

In case the universal cover of M contains some Euclidean factors, then we have the following result which is also due to Fangyang Zheng [54].

Corollary 5.1. *Let M be a complete Kähler manifold with nonnegative sectional curvature. Then its universal cover is of the form $\widetilde{M} = \mathbb{C}^k \times \widetilde{N} \times \widetilde{L}$ where $\widetilde{N}$ is a compact Hermitian symmetric manifold, $\widetilde{L}$ is Stein and $\widetilde{L}$ contains no Euclidean factor. Moreover, there exists a discrete subgroup $\Gamma \subseteq I_h(\mathbb{C}^k)$ which acts freely on $\mathbb{C}^k$, and group homomorphisms $\rho : \Gamma \to I_h(\widetilde{N})$, $\tau : \Gamma \to I_h(\widetilde{L})$, such that M is holomorphically isometric to the quotient of $\widetilde{M}$ by Γ which acts on $\widetilde{M}$ as*

$$\gamma(x, y, z) = (\gamma(x), \rho(\gamma)(y), \tau(\gamma)(z))$$

for any $\gamma \in \Gamma$. In particular, M is a holomorphic and Riemannian fiber bundle with fiber $\widetilde{N} \times \widetilde{L}$ over the flat Kähler manifold $\mathbb{C}^k/\Gamma$. Here $I_h(X)$ denotes the group of isometric biholomorphisms of a Kähler manifold X.

Proof. By Theorems 4.2 and 4.3, it is easy to see that $\widetilde{M}$ is of the form as described in the corollary. Note that $\widetilde{N}$ or $\mathbb{C}^k$ may reduce to a point. Let G be the fundamental group of M. Let $\beta \in G$, we claim that

$$\beta(x, y, z) = (f(x), g(y), h(z))$$

for $(x, y, z) \in \mathbb{C}^k \times \widetilde{N} \times \widetilde{L}$. denote the point of $\widetilde{M}$ according to the splitting. A priori $f = f(x, y, z)$. So are g and h. As in the proof of Lemma 5.1, we have $f = f(x, z)$ and $h = h(x, z)$ are independent of y since $\widetilde{N}$ is a compact Hermitian symmetric manifold. The next observation that h is also independent of x. Otherwise, there exists x_1, x_2 in $\mathbb{C}^k$ and $z \in \widetilde{L}$ such that $h(x_1, z) \neq h(x_2, z)$. Denote the line passing x_1 and x_2 in $\mathbb{C}^k$ to be $\alpha(s)$. Then $\beta(\alpha)$ is also a line. This in particular implies that $h(\alpha)$ is also a geodesic and distance realizing, therefore a line, since $h(\alpha)$ is not a point. This contradicts the assumption that

$\widetilde{L}$ does not contain any lines. Hence $h = h(z)$. As in the proof of Lemma 5.1, we conclude that $f = f(x)$. Moreover, it is easy to see that $f \in I_h(\mathbb{C}_k)$, $g \in I_h(\widetilde{N})$ and $h \in I_h(\widetilde{L})$.

Let $\rho_1 : G \to I_h(\mathbb{C}^k)$ be the homomorphism defined by the above correspondence $\beta \to f$. Define ρ_2, ρ_3 similarly. We claim that ρ_1 is a monomorphism. Otherwise, we can find $\beta \neq$identity in G such that $f =$identity, where $\beta = (f, g, h)$. Then (f, g) will act freely on $\widetilde{N} \times \widetilde{L}$. This implies that the group generated by (f, g) will be the fundamental group of some complete Kähler manifold which is covered by $\widetilde{N} \times \widetilde{L}$ and is noncompact by [19] or [4] because $\widetilde{N} \times \widetilde{L}$ contains no Euclidean factors. This contradicts Theorem 5.2. Therefore we know that ρ_1 is an isomorphism.

Now simply denote $\Gamma = \rho_1(\widetilde{\pi})$. Let $\rho = \rho_2 \circ \rho_1^{-1}$ and $\tau = \rho_3 \circ \rho_1^{-1}$. This completes the proof of the corollary. q.e.d.

6. Poincaré-Lelong equation and a gap theorem

In this section, we shall solve the Poincaré-Lelong equation using some refined estimates developed in previous sections, in particular in §1–§3. One of the motivation is to discuss the curvature decay condition (FCD) defined in Section 4, see Remark 4.2. We shall prove the following:

Theorem 6.1. *Let M^m be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let ρ be a real closed $(1,1)$ form with trace f. Suppose $f \geq 0$ and ρ satisfies the following conditions:*

$$\int_0^\infty ⨍_{B_o(s)} ||\rho|| ds < \infty, \tag{6.1}$$

and

$$\liminf_{r\to\infty} \left[\exp\left(-ar^2\right) \int_{B_o(r)} ||\rho||^2 \right] < \infty \tag{6.2}$$

for some $a > 0$. Then there is a solution u of the Poincaré-Lelong *equation* $\sqrt{-1}\partial\bar{\partial}u = \rho$. *Moreover, for any* $0 < \epsilon < 1$, u *satisfies*

$$\begin{aligned}(6.3)\qquad &\alpha_1 r \int_{2r}^{\infty} k(s)ds + \beta_1 \int_0^{2r} sk(s)ds \\ &\geq u(x) \\ &\geq -\alpha_2 r \int_{2r}^{\infty} k(s)ds - \beta_2 \int_0^{\epsilon r} sk(x,s)ds + \beta_3 \int_0^{2r} sk(s)ds\end{aligned}$$

for some positive constants $\alpha_1(m)$, $\alpha_2(m,\epsilon)$ *and* $\beta_i(m)$, $1 \leq i \leq 3$, *where* $r = r(x)$. *Here* $k(x,s) = \fint_{B_x(s)} f$ *and* $k(s) = k(o,s)$, *where* $o \in M$ *is a fixed point.*

The theorem was first proved in [30] under the assumption that M has maximal volume growth and $||\rho||(x)$ decays like $r^{-2}(x)$ pointwisely. Later in [34, Theorem 5.1] the theorem was generalized by assuming the following condition instead of (6.2):

$$\liminf_{r\to\infty} \fint_{B_o(r)} ||\rho||^2 = 0. \tag{6.4}$$

(6.4) is obviously much stronger than (6.2). However, it would be nice if (6.2) can be totally removed.

We shall use the ideas in [30] and [34]. By [34, Theorem 1.1 and Theorem 1.2], there exists a solution to the Poisson equation $\Delta u = f$ such that u satisfies (6.3). The difficult part is to prove that $\sqrt{-1}\partial\bar{\partial}u = \rho$. As in [34], one can prove that $||\sqrt{-1}\partial\bar{\partial}u||$ behaves like $||\rho||$ in the average sense. If (6.4) is assumed, the result will follow easily by using the mean value theorem of [25, p. 287]. If we only assume (6.2), the method does not work because the average of $||\rho||$ might grow exponentially.

The outline of the proof of $\sqrt{-1}\partial\bar{\partial}u = \rho$ is as follows. First we solve the Cauchy problem (1.6) with initial data $u(x)$ for all time and let $v(x,t)$ be the solution. Let $w = u - v$. By an argument as in Lemma 2.1, one can show that $\|\rho - \sqrt{-1}\partial\bar{\partial}w\|$ is a subsolution of the heat equation, and that $\|\rho - \sqrt{-1}\partial\bar{\partial}w\|(x,t) \to 0$ as $t \to \infty$ using (6.1) and (6.2). On the other hand, we shall prove that $v(x,t) - v(o,t)$ together with its derivatives uniformly converges to a constant over any fixed compact subset, at least subsequentially. Therefore $\|\sqrt{-1}\partial\bar{\partial}v\|(x,t) \to 0$, which implies that $\rho - \sqrt{-1}\partial\bar{\partial}u \equiv 0$.

As in [34], by taking $M \times \mathbb{C}^2$, we may assume that M is nonparabolic and its Green's function $G(x,y)$ satisfies the following with $n = 2m$

being the real dimension of M:

$$\sigma^{-1}\frac{r^2(x,y)}{V_x(r(x,y))} \le G(x,y) \le \sigma\frac{r^2(x,y)}{V_x(r(x,y))}$$

for some $\sigma = \sigma(n) > 0$.

As mentioned above, by [34] we can solve $\Delta u = f$ with u satisfying (6.3). u is given by

$$u(x) = \int_M \left(G(o,y) - G(x,y)\right) f(y)dy. \tag{6.5}$$

The details of the proof that $\sqrt{-1}\partial\bar{\partial}u = \rho$ are contained in the following two lemmas.

Lemma 6.1.

(i) *The Cauchy problem (1.6) with initial value u has long time solution $v(x,t)$ which is given by*

$$v(x,t) = \int_M H(x,y,t)u(y)dy.$$

(ii) *There exists $t_i \to \infty$ such that $v(x,t_i) - v(o,t_i)$ together with their derivatives converge uniformly on compact subsets to a constant function.*

Proof. (i) We want to apply Lemma 1.2. For any $R > 0$ and $x \in B_o(R)$,

$$\begin{aligned} |u(x)| &\le \int_M |G(o,y) - G(x,y)|\, f(y)dy \\ &= \left\{\int_{M\setminus B_o(4R)} + \int_{B_o(4R)}\right\} |G(o,y) - G(x,y)|\, f(y)dy \\ &= I(x) + II(x). \end{aligned} \tag{6.6}$$

By (6.1), we have

$$I(x) \le C_1 r(x)\int_{2R}^{\infty} k(s)ds \le C_2 r(x) \tag{6.7}$$

as in [34, (1.4)], for some constants C_1 and C_2 independent of R. On the other hand, as in [34, p. 347 and p. 356], we have

$$\begin{aligned}\int_{B_o(R)} II(x)\,dx &\le \int_{x\in B_o(r)}\left[\int_{y\in B_o(4R)}(G(o,y)+G(x,y))\,f(y)dy\right]dx\\ &= \int_{y\in B_o(4R)}\left[\int_{x\in B_o(R)}(G(o,y)+G(x,y))\,dx\right]f(y)dy\\ &= C_3V_o(R)\left(R^2k(4R)+\int_0^{4R}sk(s)ds\right)\end{aligned}$$

where C_3 depends only on n. Combining this with (6.6) and (6.7) and using (6.1), we conclude that

$$\fint_{B_o(R)}|u|\le C(1+R^2)$$

for some constant independent of R. By Lemma 1.2, (i) follows.

(ii) Let us first give an estimate of $|\nabla v|$. We cannot use the same method as in Lemma 1.5, because we have neither the estimate of the integral of u^2 as in Lemma 1.5, nor the uniform bound on $|\nabla u|^2$. However, we may proceed as in the proof of Lemma 1.2. Namely, use cutoff functions φ_i and denote $f_i=\varphi_i f$. Solve $\Delta u_i=f_i$ by using (6.5) and find solution v_i of (1.6) with initial value u_i. Then v_i subconverge to v together with their derivatives uniformly on compact sets of $M\times[0,\infty)$. Note that $|\nabla u_i|$ is bounded by [34, Theorem 1.3] and hence $|\nabla v_i|$ is bounded by Lemma 1.5 or [26]. We can apply the maximum principle to $|\nabla v_i|$ which is a subsolution of the heat equation and conclude that for any x such that $r(x)\le\sqrt{t}$,

$$\begin{aligned}|\nabla v_j|(x,t) &\le \int_M H(x,y,t)|\nabla u_i|(y)\,dy\\ &\le C_4\sup_{r\ge\sqrt{t}}\fint_{B_x(r)}|\nabla u_i|(y)\,dy\\ &\le C_4\sup_{r\ge\sqrt{t}}\fint_{B_o(2r)}|\nabla u_i|(y)\,dy\\ &\le C_5\sup_{r\ge\sqrt{t}}\left(\int_{4r}^\infty k(s)\,ds+rk(4r)\right)\\ &\le C_6\int_{4\sqrt{t}}^\infty k(s)\,ds\end{aligned}\tag{6.8}$$

for some constants $C_4 - C_6$ depending only on n. Here we have used Corollary 3.2 of [33] in the second inequality, Theorem 1.3 of [34] in the fourth inequality and we have also used the volume comparison as well as the fact that $0 \le f_i \le f$. Hence

$$\sup_{x\in B_o(\sqrt{t})} |\nabla v_i|(x,t) \le C_6 \int_{4\sqrt{t}}^{\infty} k(s)\, ds.$$

for all i and so

$$\sup_{x\in B_o(\sqrt{t})} |\nabla v|(x,t) \le C_6 \int_{4\sqrt{t}}^{\infty} k(s)\, ds. \tag{6.9}$$

On the other hand, f_i has compact support, u_i and v_i are bounded. Since $(v_i)_t$ is a solution to the heat equation with initial value f_i, as in the proof of Lemma 1.5 (or (6.12) below), one can prove that for any $T > 0$, there exist constants C_i such that

$$\int_0^T \fint_{B_o(r)} |(v_i)_t|^2 \le C_i$$

for all r. Hence we can apply maximum principle and conclude that

$$\begin{aligned}\frac{\partial v_i}{\partial t}(x,t) &= \int_M H(x,y,t) f_i(y)\, dy\\ &\le C(n) \sup_{r\ge\sqrt{t}} \fint_{B_x(r)} f_i(y)\, dy\\ &\le C(n) \sup_{r\ge\sqrt{t}} k(x,r).\end{aligned}$$

Here we also have used Corollary 3.2 of [33]. Note we also have $(v_i)_t \ge 0$. Hence we have

$$0 \le \frac{\partial v}{\partial t}(x,t) \le C(n) \sup_{r\ge\sqrt{t}} k(x,r) \le \frac{C(n)}{\sqrt{t}} \int_{\sqrt{t}}^{\infty} k(x,s)\, ds. \tag{6.10}$$

By (6.9), (6.10) and (6.1), for any $t_0 > 1$, and $r > 0$, the function $v(x,t) - v(o,t_0)$ is bounded in $B_o(r) \times [t_0 - 1, t_0 + 1]$ by a constant which is independent of t_0 and $\lim_{t\to\infty} \sup_{B_o(r)} |\nabla v(\cdot,t)| \to 0$. Hence, it is easy to see that (ii) is true. q.e.d.

Now let $w = u - v$.

Lemma 6.2. *As $t \to \infty$, $||\rho - \sqrt{-1}\partial\bar{\partial}w||(x,t)$ converges to zero uniformly on compact subsets in M.*

Proof. We claim that

$$\|\rho - \sqrt{-1}\partial\bar{\partial}w\|(x,t) \leq \int_M H(x,y,t)\|\rho\|(y)\,dy. \tag{6.11}$$

If (6.11) is true, then one can apply Corollary 3.2 of [33] again to conclude that, for $x \in B_o(\sqrt{t})$,

$$\begin{aligned}\|\rho - \sqrt{-1}\partial\bar{\partial}w\|(x,t) &\leq C(n)\sup_{r\geq\sqrt{t}} ⨍_{B_x(r)} \|\rho\|(y)\,dy \\ &\leq C(n)\sup_{r\geq\sqrt{t}} ⨍_{B_o(2r)} \|\rho\|(y)\,dy\end{aligned}$$

for some constant $C(n)$ depending only on n. From the assumption (6.1), this implies that $\sup_{B_o(\sqrt{t})}\|\rho - \sqrt{-1}\partial\bar{\partial}w\|(\cdot,t) \to 0$ as $t \to \infty$ and the lemma follows.

To prove (6.11), we first observe that since ρ is real d-closed $(1,1)$ form, locally it can be written as $\sqrt{-1}\partial\bar{\partial}\Psi$. Using $\Delta\Psi = f$, it is easy to see that $\Psi - w$ satisfies the heat equation. Hence $\eta = \rho - \sqrt{-1}\partial\bar{\partial}w = \sqrt{-1}\partial\bar{\partial}(\Psi - w)$ satisfies (2.1) by Lemma 2.1 in [37]. (6.11) will follow from Lemma 2.1 provided η satisfies (2.2) and (2.3). By (6.1), since $w \equiv 0$ at $t = 0$, it is easy to see that η satisfies (2.2) for any $a > 0$. Next, we estimate $|\nabla^2 v|^2$. Again, we may obtain the estimates for v_i first and let $i \to \infty$. Hence, as in the proof of Lemma 3.1, for any $T > 1$

and $r^2 \geq T$, using the first inequality in (6.8) one can prove that

$$\begin{aligned}
(6.12)\quad & \int_0^T \fint_{B_o(r)} |\nabla^2 v|^2 \\
& \leq C_1 \left[\frac{1}{r^2} \int_0^T \fint_{B_o(2r)} |\nabla v|^2 + \fint_{B_o(2r)} |\nabla u|^2 \right] \\
& \leq C_2 \left[(T+1) \fint_{B_o(8r)} |\nabla u|^2(x) dx \right. \\
& \qquad \left. + \int_0^T t^{-2} \left(\int_{8r}^\infty \exp\left(-\frac{s^2}{20t}\right) s \fint_{B_o(s)} |\nabla u|(y)\, dy ds \right)^2 dt \right] \\
& \leq C_3 \left[(T+1) \fint_{B_o(8r)} |\nabla u|^2(x) dx \right. \\
& \qquad \left. + \int_0^T \left(\int_{4r}^\infty \exp\left(-\frac{s^2}{20t}\right) d\left(\frac{s^2}{t}\right) \right)^2 \right] dt \\
& \leq C_4 (T+1) \left[\fint_{B_o(4r)} |\nabla u|^2(x) dx + 1 \right] \\
& \leq C_5 (T+1) \left[\left(\int_{16r}^\infty k(s) ds \right)^2 + r^2 \fint_{B_o(16r)} ||\rho||^2 + 1 \right] \\
& \leq C_6 (T+1) \left[r^2 \fint_{B_o(8r)} ||\rho||^2 + 1 \right]
\end{aligned}$$

for some constants $C_1 - C_6$ independent of r and T. Here we have used Lemma 1.1 in the second inequality, (6.1) and Theorem 1.3 of [34] in third inequality, Theorem 1.3 of [34] in the fifth inequality. By Theorem 1.3 in [34] again, we have

$$\fint_{B_o(r)} |\nabla^2 u|^2 \leq C_7 \left[r^2 \fint_{B_o(2r)} ||\rho||^2 + 1 \right]$$

for some constant C_7 depending only on n. Combining this with (6.12), we conclude that

$$\int_0^T \fint_{B_o(r)} ||\rho - \sqrt{-1}\partial\bar{\partial} w||^2 \leq C_7 (T+1) \left[r^2 \fint_{B_o(2r)} ||\rho||^2 + 1 \right]$$

for some constant C independent of T and r. Hence by (6.2), η also satisfies (2.3), and we can apply Lemma 2.1 to conclude that (6.11) is true. q.e.d.

Now Theorem 6.1 follows from Lemmas 6.1 and 6.2.

Using Theorems 6.1 and 3.2, we can prove that under a mild condition, the scalar curvature of a nonflat complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature cannot decay faster than (FCD).

Corollary 6.1. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and let $\rho \geq 0$ be a d-closed real $(1,1)$ form. Assume that ρ satisfies (6.2). Then $\rho \equiv 0$, if*

$$\int_0^r s\left(\fint_{B_o(s)} \|\rho\|(y)\,dy\right) ds = o(\log r). \tag{6.13}$$

In particular, if the scalar curvature $\mathcal{R}$ of M satisfies (6.2) and (6.13) with $||\rho||$ replaced by $\mathcal{R}$, then M must be flat.

Proof. By Theorem 6.1, one can solve $\sqrt{-1}\partial\bar{\partial}u = \rho$ where u satisfies (6.3). (6.13) then implies that $u(x) = o(\log r)$ and u must be constant by Theorem 3.2. Hence $\rho \equiv 0$. The last result follows from this by considering ρ to be the Ricci form of M. q.e.d.

Remark 6.1. The gap theorem in the last part of the corollary was first obtained in [30] under the assumptions that M has maximum volume growth with curvature decays like $r^{-2-\epsilon}$ pointwisely. These implies (6.4) and (6.13) are true uniformly for all $o \in M$ (with $||\rho||$ replaced by $\mathcal{R}$). Later, using Kähler-Ricci flow of [39], it was generalized in [6] by only assuming that (6.13) holds uniformly for all $o \in M$. In order to use the Kähler Ricci flow, it was also assumed that $\mathcal{R}$ is bounded in [6]. It is easy to see that if $\mathcal{R}$ is bounded, then (6.13) will imply (6.4). In Corollary 6.1, $\mathcal{R}$ might grow exponentially. Moreover, we only need to assume that (6.13) holds at one point.

In Theorem 4.3(ii), it is proved that if M has nonnegative holomorphic bisectional curvature which is positive at some point and if M supports a nonconstant polynomial growth holomorphic function, then it satisfies (FCD). The following result is a partial converse of this.

Corollary 6.2. *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose M satisfies the condition (FCD) and suppose the Ricci curvature of M is positive at a point. If the scalar curvature $\mathcal{R}$ also satisfies (6.2), then M supports nonconstant holomorphic functions of polynomial growth.*

Moreover, if Ric$(o) > 0$*, there exists* $\{f_1, \cdots, f_m\}$*, holomorphic functions of polynomial growth such that they form a local coordinates near* o*. In particular, there exists a positive constant* δ *independent of* d *such that*

$$\dim(\mathcal{O}_d(M)) \geq \delta d^m$$

for d *sufficient large. Here* $\mathcal{O}_d(M)$ *is the vector space consisting all holomorphic functions* f *such that* $|f(x)| \leq C(r(x)+1)^d$ *for some constant* $C(f)$*.*

Proof. By Theorem 6.1, we can solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \mathrm{Ric}$, since (FCD) implies $\|\mathrm{Ric}\|$ satisfies (6.1). Moreover, by (6.3) we know that $u(x)$ satisfies $u(x) \leq C\log(r(x)+2)$ for some C. The corollary then follows from rather standard argument, see [28, 32] for example. In fact, let $\{z_1, \cdots, z_m\}$ be the local coordinate near o. Let $h_i = \varphi(x)z_i$, where $\varphi(x)$ is a cut-off function which has support inside the local coordinate neighborhood. Let $\theta_i = \bar{\partial}h_i$. Now apply Theorem 3.2 in [32], with E being the anti-canonical line bundle. We then have functions η_i such that $\bar{\partial}\eta_i = \theta_i$ and $\eta_i(o) = 0$. Moreover the η_i satisfies the following estimate:

$$\int_M |\eta_i|^2 \exp(-Cu(x)) < \infty. \tag{6.14}$$

It is easy to see that $f_i = \theta_i - \eta_i$ will be holomorphic functions such that $f_i = z_i$ near o. Moreover f_i satisfies (6.14). Applying the mean value inequality of [25, p. 287] as in the proof of Lemma 4.2 we conclude that f_i are of polynomial growth. The second claim of the corollary follows from simple dimension counting. q.e.d.

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